STRESS CONCENTRATIONS IN INFINITE ORTHOTROPIC PLATES

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CHAPTER 1

INTRODUCTION

Any material which has three mutually perpendicular planes of elastic symmetry is said to be orthotropic. If from such a material a thin flat rectangular plate of uniform thickness is cut such that the plane of the plate is parallel to a plane of elastic symmetry, then this plate will have two perpendicular axes of symmetry in the plane of the plate. Such a plate is said to be orthotropic. In this dissertation the state of stress in a large rectangular plate of this type containing a circular hole filled by an isotropic material is considered. The problem is an important one. As an illustration, the problem of a wood plate containing a circular knot can be cited since both plain-sawn or quarter-sawn boards[1] are examples of orthotropic plates. In the discussion the large plate will be assumed to be subjected to a uniform tension S at large distance from the isotropic disk and directed parallel to

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[1]Alvin Jewel Owens, "Effect of a Rigid Elliptic Disk on the Stress Distribution in an Orthotropic Plate." Doctor's thesis, University of Florida, 1950, p. 2.

an axis of elastic symmetry. The stress functions for the orthotropic plate and for the isotropic disk will be obtained and an examination will be made of the stresses near the disk. It is assumed that the strains always remain within the limits of perfect elasticity, i.e. the material always returns to its initial shape after the external loading is removed.

For the orthotropic plate mentioned above with no holes and with a uniform tension applied along two opposite edges there are no stress concentrations. This means that the ratio of the normal stress at any point in the direction of the applied tension to the normal stress at infinity is one; the ratio of the normal stress at any point in the direction perpendicular to the applied tension at infinity is zero; and the ratio of the shear stresses at any point in either of the above directions to the normal stress at infinity equals zero. As has been stated the values of the ratios given above are constant for all points of the plate so that there are no stress concentrations anywhere.

When a disk is inserted in a hole in an infinite orthotropic plate and a uniform tension is applied along opposite edges of the plate as in the present case, then the ratios of the stresses at a point to the normal stress at infinity vary from point to point. There are definite stress concentrations in the neighborhood of the disk, that is, the ratios of the

stresses at some points in the vicinity of the disk to the normal stress at infinity are larger than the corresponding ratios for some other points in the plane. An examination of the stresses in the vicinity of the disk will be made with the idea in mind of determining the points of greatest stress concentrations.

Dr. A. J. Owens[2] treated the problem of an infinite orthotropic plate with a small elliptic hole at the center filled with a rigid elliptic disk. In this problem a circular hole in an infinite orthotropic plate is filled with a circular, non-rigid disk of an isotropic material.

Professor C. B. Smith[3] treated the problem of an infinite rectangular orthotropic plate with a small elliptic or circular hole at its center. Professor Smith found greater stress concentrations near the free hole than Dr. Owens did near the rigid elliptic disk. For this problem the stress concentrations are found to be intermediate between those found by Professor Smith and those found by Dr. Owens.

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[2]Owens, op. cit.

[3]C. B. Smith, Effect of Elliptic or Circular Holes on the Stress Distribution in Plates of Wood or Plywood Considered as Orthotropic Materials, Forest Products Laboratory Report No. 1510, 1944.

CHAPTER II

SOLUTION FOR PLANE STRESS

The center of the circular disk of radius a is chosen as the origin of the coordinate system. It is assumed that the hole is at the center of the plate(whenever the word plate is used the region exterior to the disk is intended). The axes are taken to lie parallel to the two axes of elastic symmetry of the plate, hence the uniform tension at infinity acts parallel to a coordinate axis. (Fig. 1)

The mathematical notation of Professor S. Timoshenko[4] will be used throughout this treatise. Using this notation the boundary conditions on the periphery of the plate are:

$$\sigma_x (x = \pm\infty) = S$$

$$\sigma_y (y = \pm\infty) = 0 \qquad (1)$$

$$\tau_{xy} \left(\begin{matrix} x = \pm\infty \\ y = \pm\infty \end{matrix} \right) = 0 \ .$$

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[4] Timoshenko, S., Theory of Elasticity (New York: McGraw-Hill Book Company, 1951).

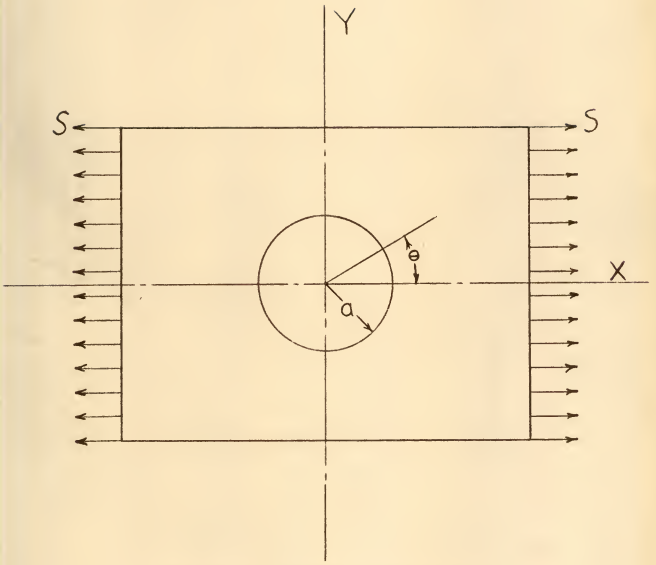

Figure 1.--Circular Isotropic Disk in an Infinite Orthotropic Plate.

The boundary conditions on the boundary of the disk are, using a prime to denote the stresses and displacements in the disk:

$$\left.\begin{aligned} \sigma_r' &= \sigma_r \\ \tau_{r\theta}' &= \tau_{r\theta} \\ u' &= u \\ v' &= v \end{aligned}\right\} \quad r = a \tag{2}$$

The plate is considered to be thin and the thickness in Z direction is taken equal to unity. Since the X- and Y- axes are the axes of elastic symmetry of the plate and since the plate is in a state of plane stress the following relations between the components of stress and strain hold:

$$\begin{aligned} e_x &= \frac{\partial u}{\partial x} = \frac{\sigma_x}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_y \\ e_y &= \frac{\partial v}{\partial y} = \frac{\sigma_y}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_x \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{G_{xy}}\tau_{xy}. \end{aligned} \tag{3}$$

In these equations E_x and E_y are Young's moduli in the X and the Y directions respectively. The symbol ν is

Poisson's ratio and ν_{xy} is the ratio of the contraction parallel to the Y-axis to the extension parallel to the X-axis associated with a tension parallel to the X-axis. The symbol G is the modulus of rigidity; G_{xy} is the modulus of rigidity associated with the directions of X and Y.

For plane stress the equations of equilibrium are:

$$\frac{\partial \sigma_x}{\partial X} + \frac{\partial \tau_{xy}}{\partial Y} = 0$$
$$\frac{\partial \tau_{xy}}{\partial X} + \frac{\partial \sigma_y}{\partial Y} = 0 . \qquad (4)$$

These equations of equilibrium are obtained from the conditions of static equilibrium without reference to the laws relating stresses and strains. They obviously will be satisfied by any stress function F such that

$$\sigma_x = \frac{\partial^2 F}{\partial Y^2}$$
$$\sigma_y = \frac{\partial^2 F}{\partial X^2} \qquad (5)$$
$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} .$$

The compatibility equation for plane stress

$$\frac{\partial^2 e_x}{\partial Y^2} + \frac{\partial^2 e_Y}{\partial X^2} = \frac{\partial^2 \gamma_{xy}}{\partial X \partial Y} \tag{6}$$

is obtained by utilizing the notion that $u(x,y)$ and $v(x,y)$, the displacements in the X and Y directions respectively, are single-valued and continuous.

Substitute equations (5) into equations (3) and obtain

$$e_x = \frac{1}{E_x}\frac{\partial^2 F}{\partial Y^2} - \frac{\nu_{yx}}{E_y}\frac{\partial^2 F}{\partial X^2}$$

$$e_Y = \frac{1}{E_y}\frac{\partial^2 F}{\partial X^2} - \frac{\nu_{xy}}{E_x}\frac{\partial^2 F}{\partial Y^2} \tag{7}$$

$$\gamma_{xy} = \frac{-1}{G_{xy}}\frac{\partial^2 F}{\partial X \partial Y}.$$

Using $\frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x}$ equations (7) become

$$e_x = \frac{1}{E_x}\frac{\partial^2 F}{\partial Y^2} - \frac{\nu_{xy}}{E_x}\frac{\partial^2 F}{\partial X^2}$$

$$e_Y = \frac{1}{E_y}\frac{\partial^2 F}{\partial X^2} - \frac{\nu_{yx}}{E_y}\frac{\partial^2 F}{\partial Y^2} \tag{8}$$

$$\gamma_{xy} = \frac{-1}{G_{xy}}\frac{\partial^2 F}{\partial X \partial Y}.$$

Substituting these values into the compatibility equation, there results

$$\frac{1}{E_x}\left[\frac{\partial^4 F}{\partial Y^4} - \nu_{xy}\frac{\partial^4 F}{\partial X^2 \partial Y^2}\right] + \frac{1}{E_y}\left[\frac{\partial^4 F}{\partial X^4} - \nu_{yx}\frac{\partial^4 F}{\partial X^2 \partial Y^2}\right] = \frac{-1}{G_{xy}}\frac{\partial^4 F}{\partial X^2 \partial Y^2}$$

$$\frac{1}{E_y}\frac{\partial^4 F}{\partial X^4} + \left[\frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x}\right]\frac{\partial^4 F}{\partial X^2 \partial Y^2} + \frac{1}{E_x}\frac{\partial^4 F}{\partial Y^4} = 0. \qquad (9)$$

Equation (9) can be written as

$$\frac{\partial^4 F}{\partial X^4} + 2K\frac{\partial^4 F}{\partial X^2 \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0 \qquad (10)$$

where[5]

$$K = \frac{\sqrt{E_x E_y}}{2}\left[\frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x}\right]$$

$$\eta = \epsilon Y$$

and $$\epsilon = \sqrt[4]{\frac{E_x}{E_y}}\ .$$

Substitution of $F = F(X + \delta\eta)$ into the new form of the compatibility equation (10) leads to the following equation in δ

$$\delta^4 + 2K\delta^2 + 1 = 0. \qquad (11)$$

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[5] H. W. March, Flat Plates of Plywood under Uniform or Concentrated Loads, Forest Products Laboratory Report No. 1312, March, 1942, p. 40.

This is a quadratic in δ^2 so that

$$\delta^2 = -K \pm \sqrt{K^2-1}\ .$$

Since for wood K as defined is probably always greater than one[5], let $K = \cosh\phi$ then

$$\delta^2 = -\cosh\phi \pm \sqrt{\cosh^2\phi - 1}$$

$$\delta^2 = -\cosh\phi \pm \sinh\phi$$

$$\delta^2 = -\frac{e^{\phi}+e^{-\phi}}{2} \pm \frac{e^{\phi}-e^{-\phi}}{2} = -e^{\phi} \text{ or } -e^{-\phi}\ .$$

Therefore the roots of equation (11) are

$$\delta_1 = ie^{\phi/2}$$
$$\delta_2 = ie^{-\phi/2}$$
$$\delta_3 = -ie^{\phi/2}$$
$$\delta_4 = -ie^{-\phi/2}\ .$$

The compatibility equation (10) can also be rewritten as

$$\left[\frac{\partial^2}{\partial x^2} + \alpha^2\frac{\partial^2}{\partial \eta^2}\right]\left[\frac{\partial^2}{\partial x^2} + \beta^2\frac{\partial^2}{\partial \eta^2}\right]F = 0 \qquad (12)$$

where $\alpha^2 = e^{\phi}$, $\beta^2 = e^{-\phi}$

since $\alpha^2 + \beta^2 = e^{\phi} + e^{-\phi} = 2\cosh\phi = 2K$.

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[6] Smith, op. cit., p. 5.

The differential equation (12) is satisfied by any stress function of the form

$$F = R\left[F_1(x+i\alpha\eta) + F_2(x+i\beta\eta)\right] \tag{13}$$

where

$$\alpha = e^{\Phi/2} = \sqrt{K+\sqrt{K^2-1}}$$

$$\beta = e^{-\Phi/2} = \sqrt{K-\sqrt{K^2-1}}\,.$$

The stress function in (13) may easily be demonstrated to be a solution of (12) by direct substitution.

Thus equations (12) and (13) are valid for the stress function associated with any problem of plane stress in an orthotropic plate with the X- and Y-axes being axes of elastic symmetry.

For the problem under consideration a stress function of the form (13) which satisfies the boundary conditions at infinity and on the boundary of the disk is sought.

Professor Smith[7] found in his problem of an infinite orthotropic plate with an elliptic or circular hole at the center that a satisfactory stress function is

$$F = R\left[\frac{A}{2\gamma_1^2}\left[\frac{1}{2}(Z_1-W_1)^2 + \gamma_1^2 \ln(Z_1+W_1)\right]\right.$$

$$\left.+\frac{B}{2\gamma_2^2}\left[\frac{1}{2}(Z_2-W_2)^2 + \gamma_2^2 \ln(Z_2+W_2)\right]\right] + \frac{SY^2}{2}$$

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[7] Smith, op. cit.

where

$$Z_1 = X + i\alpha\epsilon Y \qquad Z_2 = X + i\beta\epsilon Y$$

$$W_1 = \sqrt{Z_1^2 - \gamma_1^2} \qquad W_2 = \sqrt{Z_2^2 - \gamma_2^2}$$

$$\gamma_1^2 = a^2 - \alpha^2\epsilon^2 b^2 \qquad \gamma_2^2 = a^2 - \beta^2\epsilon^2 b^2 .$$

It is recalled that $\eta = \epsilon Y$. Also the boundary of the hole is given by

$$X = a\cos\theta$$
$$Y = b\sin\theta$$

and a uniform tension is applied at the ends of the plate.

Owens[8] used this same stress function for his problem which has been mentioned previously. Since Owens' stress function satisfied the boundary conditions at infinity it seems likely that his stress function will be suitable for this problem if it satisfies the boundary conditions on the periphery of the disk. It is found that his stress function does satisfy the imposed boundary conditions.

The stress function then is

$$F = R\left[\frac{A}{2\gamma_1^2}\left[\frac{1}{2}(Z_1 - W_1)^2 + \gamma_1^2 \ln(Z_1 + W_1)\right]\right.$$
$$\left. + \frac{B}{2\gamma_2^2}\left[\frac{1}{2}(Z_2 - W_2)^2 + \gamma_2^2 \ln(Z_2 + W_2)\right]\right] + \frac{SY^2}{2} \qquad (14)$$

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[8] Owens, op. cit., p. 15.

where $A = A_1 + iA_2$ $\qquad B = B_1 + iB_2$

$$Z_1 = X + i\alpha\epsilon Y \qquad Z_2 = X + i\beta\epsilon Y$$

$$W_1 = \sqrt{Z_1^2 - \gamma_1^2} \qquad W_2 = \sqrt{Z_2^2 - \gamma_2^2} \tag{15}$$

but where now

$$\gamma_1^2 = a^2(1-\alpha^2\epsilon^2) \qquad \gamma_2^2 = a^2(1-\beta^2\epsilon^2).$$

The boundary of the disk is given by

$$X = a\cos\theta$$
$$Y = a\sin\theta$$

and S is the uniform tension applied to the ends of the plate in the X direction.

To insure that the stresses be single-valued the square roots in (15) are so chosen that the inequalities

$$|Z_1 + W_1| \geq |\gamma_1|$$
$$|Z_2 + W_2| \geq |\gamma_2|$$

always hold. After the stresses are obtained it will be more evident why such restrictions are necessary. Therefore after the expressions for the stresses are obtained these inequalities will be discussed further.

For convenience of reference a list of useful relations will be given here.

$$\frac{\partial f_1(z_1)}{\partial X} = \frac{\partial f_1}{\partial z_1}\frac{\partial z_1}{\partial X} = \frac{\partial f_1}{\partial z_1}$$

$$\frac{\partial f_2(z_2)}{\partial X} = \frac{\partial f_2}{\partial z_2}\frac{\partial z_2}{\partial X} = \frac{\partial f_2}{\partial z_2}$$

$$\frac{\partial f_1(z_1)}{\partial Y} = \frac{\partial f_1}{\partial z_1}\frac{\partial z_1}{\partial Y} = i\alpha\epsilon\frac{\partial f_1}{\partial z_1}$$

$$\frac{\partial f_2(z_2)}{\partial Y} = \frac{\partial f_2}{\partial z_2}\frac{\partial z_2}{\partial Y} = i\beta\epsilon\frac{\partial f_2}{\partial z_2}$$

$$\frac{\partial w_1}{\partial z_1} = \frac{z_1}{\sqrt{z_1^2-\gamma_1^2}} = \frac{z_1}{w_1}$$

$$\frac{\partial w_2}{\partial z_2} = \frac{z_2}{\sqrt{z_2^2-\gamma_2^2}} = \frac{z_2}{w_2}.$$

From the stress function, using the above relations where necessary, one has

$$\frac{\partial F}{\partial X} = R\left[\frac{A}{2\gamma_1^2}\left[(z_1-w_1)\left(1-\frac{z_1}{w_1}\right)+\gamma_1^2\frac{1+\frac{z_1}{w_1}}{z_1+w_1}\right]\right.$$
$$\left.\frac{B}{2\gamma_2^2}\left[(z_2-w_2)\left(1-\frac{z_2}{w_2}\right)+\gamma_2^2\frac{1+\frac{z_2}{w_2}}{z_2+w_2}\right]\right]$$

$$\frac{\partial F}{\partial X} = R\left[\frac{A}{2\gamma_1^2}\left[\frac{-(z_1-w_1)^2}{w_1}+\frac{\gamma_1^2}{w_1}\right]+\frac{B}{2\gamma_2^2}\left[\frac{-(z_2-w_2)^2}{w_2}+\frac{\gamma_2^2}{w_2}\right]\right].$$

Rewrite as follows

$$\frac{\partial F}{\partial x} = R\left[\frac{A}{2\gamma_1^2}\left[\frac{-(Z_1-W_1)(Z_1^2-W_1^2)}{W_1(Z_1+W_1)} + \frac{\gamma_1^2(Z_1+W_1)}{W_1(Z_1+W_1)}\right] + \frac{B}{2\gamma_2^2}\left[\frac{-(Z_2-W_2)(Z_2^2-W_2^2)}{W_2(Z_2+W_2)} + \frac{\gamma_2^2(Z_2+W_2)}{W_2(Z_2+W_2)}\right]\right].$$

Since

$$W_1 = \sqrt{Z_1^2 - \gamma_1^2}$$

then $Z_1^2 - W_1^2 = \gamma_1^2$.

Using this relation $\frac{\partial F}{\partial X}$ becomes

$$\frac{\partial F}{\partial X} = R\left[\frac{A}{2}\left[\frac{-(Z_1-W_1)+(Z_1+W_1)}{W_1(Z_1+W_1)}\right] + \frac{B}{2}\left[\frac{-(Z_2-W_2)+(Z_2+W_2)}{W_2(Z_2+W_2)}\right]\right]$$

or

$$\frac{\partial F}{\partial X} = R\left[\frac{A}{Z_1+W_1} + \frac{B}{Z_2+W_2}\right].$$

$$\sigma_Y = \frac{\partial^2 F}{\partial X^2} = R\left[A\left[\frac{-\left(1+\frac{Z_1}{W_1}\right)}{(Z_1+W_1)^2}\right] + B\left[\frac{-\left(1+\frac{Z_2}{W_2}\right)}{(Z_2+W_2)^2}\right]\right].$$

But this gives

$$\sigma_Y = R\left[\frac{-A}{W_1(Z_1+W_1)} + \frac{-B}{W_2(Z_2+W_2)}\right]. \qquad (16)$$

It now becomes apparent what the form of the expression for the other stresses will be. Performance of the necessary differentiation yields,

$$\sigma_X = \frac{\partial^2 F}{\partial Y^2} = R\left[\frac{\alpha^2\epsilon^2 A}{W_1(Z_1+W_1)} + \frac{\beta^2\epsilon^2 B}{W_2(Z_2+W_2)}\right] + S \qquad (17)$$

and

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = R\left[\frac{i\alpha\epsilon A}{W_1(Z_1+W_1)} + \frac{i\beta\epsilon B}{W_2(Z_2+W_2)}\right]. \qquad (18)$$

The stress formulas (16), (17), and (18) contain multiple-valued expressions

$$\frac{1}{W_1(Z_1+W_1)} \quad \text{and} \quad \frac{1}{W_2(Z_2+W_2)}.$$

Assume that $\gamma_1^2 < 0$

and let $\gamma_1^2 = -C^2$.

Define

$$W = Z_1 + W_1 = Z_1 \pm \sqrt{Z_1^2 + C^2}.$$

The branch points of the function W are readily seen to be $\pm$ iC. Since there are two and only two branch points the branch cut may be taken as a straight line joining the points iC and -iC. How does this branch cut map onto the W-plane?

Given $Z_1 = X + i\alpha\epsilon Y$

$$X_1 = x = 0$$

and $$Y_1 = \alpha\epsilon Y,$$

along the branch cut

$$W = U + iV = Z_1 \pm \sqrt{Z_1^2 + C^2}$$

$$W = iY_1 \pm \sqrt{C^2 - Y_1^2}$$

where $|\gamma_1| \leq |C|$;

whereby

$$u = \pm\sqrt{C^2 - \gamma_1^2} \quad ; \quad v = \gamma_1 .$$

$u^2 + v^2 = \gamma_1^2 + C^2 - \gamma_1^2 = C^2$ is a circle with center at the origin in the W plane and of raduis $|C|$.

If we restrict W such that

$$|W| = |Z_1 + W_1| \geq |C| = |\gamma_1|$$

then the variable Z_1 is prohibited from following a path which crosses the branch cut and thus W remains single-valued. The entire Z_1 plane with the exception of the branch cut is mapped into the region in the W plane exterior to the circle at the origin of radius $|C|$.

Assume $\gamma_1^2 > 0$, then

$$W = Z_1 + W_1 = Z_1 \pm\sqrt{Z_1^2 - \gamma_1^2}$$

Now the branch points of W are $\pm\gamma_1$, and a logical branch cut is a straight line joining the points γ_1 and $-\gamma$, in the Z_1- plane. The branch cut maps into a circle in the W- plane with center at the origin and of radius γ_1 . The restriction used before

$$|W| = |Z_1 + W_1| \geq |\gamma_1|$$

insures that the branch cut will not be crossed and that W will remain single-valued.

Obviously an entirely analogous argument relative to Z_2, W_2, and γ_2 yields the relation

$$|z_2+w_2| \geq |\gamma_2|$$

in order that the stresses remain single-valued.

It is necessary to show that the given stress function yields stresses which satisfy the boundary conditions at infinity. When $X = \pm\infty$ then $Z_1 = Z_2 = \infty$, and $Z_1+W_1 = Z_2+W_2 = \infty$.

Examining equations (17), and (18), it is found that

$$\sigma_x \begin{bmatrix} x\to\infty \\ x\to-\infty \end{bmatrix} = R[0+0] + S = S$$

provided that the constants A and B are not infinite. Likewise

$$\tau_{xy} \begin{bmatrix} x\to\infty \\ x\to-\infty \end{bmatrix} = R[0+0] = 0.$$

For the two edges of the plate $Y = \pm\infty$, $Z_1 = Z_2 = \infty$ and $Z_1+W_1 = Z_2+W_2 = \infty$.

Equations (16) and (18) yield

$$\sigma_y \begin{bmatrix} Y\to\infty \\ Y\to-\infty \end{bmatrix} = R[0+0] = 0$$

$$\tau_{xy} \begin{bmatrix} Y\to\infty \\ Y\to-\infty \end{bmatrix} = R[0+0] = 0.$$

Hence it is seen that all the boundary conditions at the edges of the plate are satisfied.

There remain then four boundary conditions to be satisfied at the periphery of the disk. These four conditions mathematically stated are:

$$\left.\begin{aligned} \sigma_r' &= \sigma_r \\ \tau_{r\theta}' &= \tau_{r\theta} \\ u' &= u \\ v' &= v \end{aligned}\right] \quad r = a \qquad (2)$$

where the primes denote the disk.

Since σ_r can be obtained from the rectangular stresses σ_x, σ_y, and τ_{xy} the relation between them will be obtained.

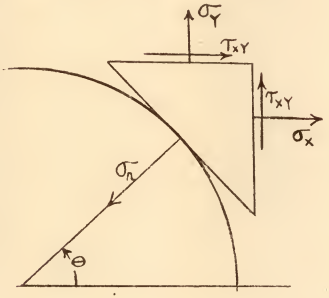

Figure 2

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy} \sin\theta \cos\theta . \qquad (19)$$

Substitution of equations (16), (17) and (18) into equation (19) yields

$$\sigma_r = R\left[\frac{A}{W_1(Z_1+W_1)}\left(\alpha^2\epsilon^2\cos^2\theta - \sin^2\theta + 2i\alpha\epsilon\sin\theta\cos\theta\right)\right.$$

$$\left. + \frac{B}{W_2(Z_2+W_2)}\left(\beta^2\epsilon^2\cos^2\theta - \sin^2\theta + 2i\beta\epsilon\sin\theta\cos\theta\right)\right]$$

$$+ S\cos^2\theta.$$

On the boundary of the disk r = a. What is the value of W_1 and $Z_1 + W_1$ when r = a?

Since X = a cos θ and y = a sin θ then

$$Z_1 + W_1 = a(\cos\theta + i\alpha\epsilon\sin\theta)$$

$$+ \sqrt{a^2(\cos^2\theta + 2i\alpha\epsilon\sin\theta\cos\theta - \alpha^2\epsilon^2\sin^2\theta) - a^2(-\alpha^2\epsilon^2)}$$

or $Z_1+W_1 = a(\cos\theta+i\alpha\epsilon\sin\theta)$

$$+a\sqrt{-\sin^2\theta + 2i\alpha\epsilon\sin\theta\cos\theta - \alpha^2\epsilon^2\sin^2\theta + \alpha^2\epsilon^2}\ .$$

Using a trigonometric identity this becomes

$$Z_1 + W_1 = a(\cos\theta + i\sin\theta)$$

$$+a\sqrt{-\sin^2\theta + 2i\alpha\epsilon\sin\theta\cos\theta + \alpha^2\epsilon^2\cos^2\theta}$$

$$Z_1+W_1 = a\left[\cos\theta + i\alpha\epsilon\sin\theta + \sqrt{(\alpha\epsilon\cos\theta + i\sin\theta)^2}\right].$$

Thus

$$Z_1+W_1 = a\left[(1+\alpha\epsilon)\cos\theta + i(1+\alpha\epsilon)\sin\theta\right]$$

or rewriting

$$Z_1+W_1 = a(1+\alpha\epsilon)(\cos\theta + i\sin\theta).$$

From the above computation it is seen that

$$W_1 = a(\alpha\epsilon\cos\theta + i\sin\theta).$$

Similarly

$$W_2 = a(\beta\epsilon\cos\theta + i\sin\theta)$$

and $Z_2+W_2 = a(1+\beta\epsilon)(\cos\theta + i\sin\theta)$.

The above values for W_1, W_2, $Z_1 + W_1$, and $Z_2 + W_2$ permit the rewriting of σ_π.

$$\sigma_\pi(\pi=a) = R\left[\frac{A(\alpha\epsilon\cos\theta + i\sin\theta)^2}{a^2(\alpha\epsilon\cos\theta + i\sin\theta)(1+\alpha\epsilon)(\cos\theta+i\sin\theta)}\right.$$

$$\left. + \frac{B(\beta\epsilon\cos\theta+i\sin\theta)^2}{a^2(\beta\epsilon\cos\theta + i\sin\theta)(1+\beta\epsilon)(\cos\theta+i\sin\theta)}\right] + S\cos^2\theta.$$

This reduces to

$$\sigma_\pi(\pi=a) = R\left[\frac{A(\alpha\epsilon\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}{a^2(1+\alpha\epsilon)}\right.$$

$$\left. + \frac{B(\beta\epsilon\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}{a^2(1+\beta\epsilon)}\right] + S\cos^2\theta.$$

Inasmuch as it will be desirable to have σ_n expressed in terms of the sines and cosines of various multiples of the angle θ, the powers and products of the sine and cosine functions will be converted.

$$\sigma_n(n=a) = R\left[\frac{(\alpha\epsilon\cos^2\theta + i(1-\alpha\epsilon)\sin\theta\cos\theta + \sin^2\theta)(A_1 + iA_2)}{a^2(1+\alpha\epsilon)}\right.$$

$$\left. + \frac{(\beta\epsilon\cos^2\theta + i(1-\beta\epsilon)\sin\theta\cos\theta + \sin^2\theta)(B_1 + iB_2)}{a^2(1+\beta\epsilon)}\right]$$

$$+ S\cos^2\theta.$$

By actually taking the real part of the above expression there results

$$\sigma_n(n=a) = \frac{A_1(\alpha\epsilon\cos^2\theta + \sin^2\theta) - A_2(1-\alpha\epsilon)\sin\theta\cos\theta}{a^2(1+\alpha\epsilon)}$$

$$+ \frac{B_1(\beta\epsilon\cos^2\theta + \sin^2\theta) + B_2(\beta\epsilon - 1)\sin\theta\cos\theta}{a^2(1+\beta\epsilon)} + S\cos^2\theta.$$

Now $$\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

These relations permit the expression of σ_n in the following form:

$$\sigma_n = \frac{A_1[\alpha\epsilon(1+\cos 2\theta+1)-\cos 2\theta]}{2a^2(1+\alpha\epsilon)} + \frac{A_2(\alpha\epsilon-1)\sin 2\theta}{2a^2(1+\alpha\epsilon)}$$

$$+ \frac{B_1[\beta\epsilon(1+\cos 2\theta)+1-\cos 2\theta]}{2a^2(1+\beta\epsilon)} + \frac{B_2(\beta\epsilon-1)\sin 2\theta}{2a^2(1+\beta\epsilon)} + \frac{S}{2} + \frac{S\cos 2\theta}{2}.$$

This can be rewritten as

$$\sigma_n = \frac{1}{2a^2(1+\alpha\epsilon)}\left[(1+\alpha\epsilon)A_1 + (\alpha\epsilon-1)A_1\cos 2\theta + (\alpha\epsilon-1)A_2\sin 2\theta\right]$$

$$+ \frac{1}{2a^2(1+\beta\epsilon)}\left[(1+\beta\epsilon)B_1 + (\beta\epsilon-1)B_1\cos 2\theta + (\beta\epsilon-1)B_2\sin 2\theta\right] \quad (20)$$

$$+ \frac{S}{2} + \frac{S\cos 2\theta}{2}.$$

Next an expression for the shearing stress $\tau_{n\theta}$ in the above form will be obtained from the rectangular stress components. Utilizing Figure 3 it is readily seen that

$$\tau_{n\theta} = -\sigma_x \sin\theta\cos\theta + \sigma_Y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta).$$

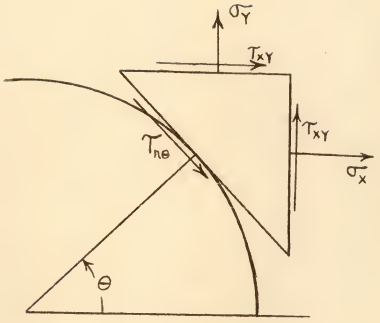

Figure 3

Use expressions (16), (17), and (18) to obtain

$$\tau_{n\theta} = R\left[\frac{-A(\cos\theta - i\sin\theta)(\alpha\epsilon^2\sin\theta\cos\theta + \sin\theta\cos\theta + i\alpha\epsilon[\sin^2\theta - \cos^2\theta])}{a^2(1+\alpha\epsilon)(\alpha\epsilon\cos\theta + i\sin\theta)}\right.$$

$$\left. - \frac{B(\cos\theta - i\sin\theta)(\beta\epsilon^2\sin\theta\cos\theta + \sin\theta\cos\theta + i\beta\epsilon(\sin^2\theta - \cos^2\theta))}{a^2(1+\beta\epsilon)(\beta\epsilon\cos\theta + i\sin\theta)}\right]$$

$$- S\sin\theta\cos\theta.$$

This will simplify somewhat by factoring to the form

$$\tau_{n\theta} = -\frac{1}{a^2(1+\alpha\epsilon)} R\left[\frac{A(\cos\theta - i\sin\theta)(\alpha\epsilon\cos\theta + i\sin\theta)(\alpha\epsilon\sin\theta - i\cos\theta)}{\alpha\epsilon\cos\theta + i\sin\theta}\right]$$

$$- \frac{1}{a^2(1+\beta\epsilon)} R \left[\frac{B(\cos\theta - i\sin\theta)(\beta\epsilon\cos\theta + i\sin\theta)(\beta\epsilon\sin\theta - i\cos\theta)}{\beta\epsilon\cos\theta + i\sin\theta} \right]$$

$$- \frac{S\sin 2\theta}{2}$$

$$\tau_{r\theta} = -\frac{1}{a^2} R \left[\frac{(A_1 + iA_2)(\alpha\epsilon\sin\theta\cos\theta - i[\alpha\epsilon\sin^2\theta + \cos^2\theta] - \sin\theta\cos\theta)}{1+\alpha\epsilon} \right.$$

$$\left. + \frac{(B_1 + iB_2)(\beta\epsilon\sin\theta\cos\theta - i(\beta\epsilon\sin^2\theta + \cos^2\theta) - \sin\theta\cos\theta)}{1+\beta\epsilon} \right] - \frac{S\sin 2\theta}{2} .$$

Now actually take the real part of the expression, and one has

$$\tau_{r\theta} = \frac{-(\alpha\epsilon - 1)\sin\theta\cos\theta A_1 - (\alpha\epsilon\sin^2\theta + \cos^2\theta)A_2}{a^2(1+\alpha\epsilon)}$$

$$+ \frac{-(\beta\epsilon - 1)\sin\theta\cos\theta B_1 - (\beta\epsilon\sin^2\theta + \cos^2\theta)B_2}{a^2(1+\beta\epsilon)} - \frac{S\sin 2\theta}{2} .$$

Apply the trigonometric identities previously used in obtaining σ_r ; it follows that

$$\tau_{r\theta} = \frac{-(\alpha\epsilon - 1)A_1\sin 2\theta - [1 + \alpha\epsilon + (1-\alpha\epsilon)\cos 2\theta]A_2}{2a^2(1+\alpha\epsilon)}$$

(21)

$$- \frac{(\beta\epsilon - 1)B_1\sin 2\theta + [1 + \beta\epsilon + (1-\beta\epsilon)\cos 2\theta]B_2}{2a^2(1+\beta\epsilon)} - \frac{S\sin 2\theta}{2} .$$

Now the polar stresses σ_r and $\tau_{r\theta}$ for the plate are obtained in the desired form. In order to satisfy the boundary conditions at the periphery of the disk it is necessary to have the corresponding stresses for the disk and for these stresses to be in the form indicated above. What is the stress function for the disk?

S. Timoshenko[9] has given the general solution for the two dimensional problem of an isotropic ring or portion of a ring, which includes as a special case the problem of the disk. This solution is given in the form of a stress function and is set forth below with the prime indicating that the expression refers to the disk.

$$F' = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 r^2\theta + a_0'\theta$$

$$+ \frac{a_1}{2} r\theta \sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r \ln r)\cos\theta$$

$$- \frac{c_1}{2} r\theta \cos\theta + (d_1 r^3 + c_1' r^{-1} + d_1' r \ln r)\sin\theta$$

$$+ \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2})\cos n\theta$$

$$+ \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2})\sin n\theta.$$

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[9] Timoshenko, op. cit., p. 114.

To prevent multiple-valued stresses take $a_0 = c_0 = d_0 = b_1' = d_1' = 0$. Here the primes in the right member do not refer to the disk.

In polar coordinates the relations between the polar stresses σ_r' and $\tau_{r\theta}'$ and the stress function are

$$\sigma_r' = \frac{1}{r}\frac{\partial F'}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F'}{\partial \theta^2}$$

$$\tau_{r\theta}' = \frac{1}{r^2}\frac{\partial F'}{\partial \theta} - \frac{1}{r}\frac{\partial^2 F'}{\partial r \partial \theta}. \qquad (22)$$

Thus $\frac{\partial F'}{\partial r}$, $\frac{\partial F'}{\partial \theta}$, $\frac{\partial^2 F'}{\partial \theta^2}$, and $\frac{\partial^2 F'}{\partial r \partial \theta}$ must be obtained. First differentiate F' with respect to r.

$$\frac{\partial F'}{\partial r} = 2b_0 r + \frac{a_1 \theta \sin\theta}{2} + (3b_1 r^2 - a_1' r^{-2})\cos\theta - \frac{c_1 \cos\theta}{2}$$

$$+ (3d_1 r^2 - c_1' r^{-2})\sin\theta$$

$$+ \sum_{n=2}^{\infty}\left(n a_n r^{n-1} + (n+2) b_n r^{n+1} - n a_n' r^{-n+1} - (n-2) b_n' r^{-n+1}\right)\cos n\theta$$

$$+ \sum_{n=2}^{\infty}\left(n c_n r^{n-1} + (n+2) d_n r^{n+1} - n c_n' r^{-n+1} - (n-2) d_n' r^{-n+1}\right)\sin n\theta.$$

Since $\frac{\partial F'}{\partial r}$ is multiplied by $\frac{1}{r}$ in the expression for σ_r' and since r ranges to zero then it is obvious that

a_1', a_1', c_1', a_n', b_n', c_n', and d_n' must equal zero to prevent infinite stresses at the origin.

Thus differentiating with respect to θ and noting that the above constants are zero, one obtains

$$\frac{\partial F'}{\partial \theta} = a_0' - b_1 r^3 \sin\theta - \frac{c_1 r}{2}(\cos\theta - \theta\sin\theta)$$

$$- d_1 r^3 \cos\theta - \sum_{n=2}^{\infty}(a_n r^n + b_n r^{n+2}) n \sin n\theta$$

$$+ \sum_{n=2}^{\infty}(c_n r^n + d_n r^{n+2}) n \cos n\theta .$$

It is further noted that $\frac{\partial F'}{\partial \theta}$ is multiplied by $\frac{1}{r^2}$ in the expression for $\tau_{r\theta}'$. To prevent an infinite stress as $r \to o$ it follows that a_0' and c_1 must be zero.

Next $\frac{\partial^2 F'}{\partial \theta^2}$ is obtained as follows

$$\frac{\partial^2 F'}{\partial \theta^2} = -b_1 r^3 \cos\theta - d_1 r^3 \sin\theta$$

$$-\sum_{n=2}^{\infty}(a_n r^n + b_n r^{n+2}) n^2 \cos n\theta$$

$$-\sum_{n=2}^{\infty}(c_n r^n + d_m r^{n+2}) n^2 \sin n\theta .$$

One other partial derivative is needed and is obtained in the following form

$$\frac{\partial^2 F'}{\partial \pi \partial \theta} = -3b_1\pi^2\sin\theta + 3d_1\pi^2\cos\theta$$

$$-\sum_{n=2}^{\infty}\left(na_n\pi^{n-1} + (n+2)b_n\pi^{n+1}\right) n\sin n\theta$$

$$+\sum_{n=2}^{\infty}\left(nc_n\pi^{n-1} + (n+2)d_n\pi^{n+1}\right) n\cos n\theta .$$

Substituting these values into equations (22) yields

$$\sigma_\pi' = 2b_0 + 3b_1\pi\cos\theta + 3d_1\pi\sin\theta$$

$$+\sum_{n=2}^{\infty}\left(na_n\pi^{n-2} + (n+2)b_n\pi^{n}\right)\cos n\theta$$

$$+\sum_{n=2}^{\infty}\left(nc_n\pi^{n-2} + (n+2)d_n\pi^{n}\right)\sin n\theta - b_1\pi\cos\theta$$

$$-d_1\pi\sin\theta - \sum\left(a_n\pi^{n-2} + b_n\pi^{n}\right)n^2\cos n\theta$$

$$-\sum_{n=2}^{\infty}\left(c_n\pi^{n-2} + d_n\pi^{n}\right)n^2\sin n\theta .$$

A simpler form would be

$$\sigma_r' = 2b_0 + 2b_1 r\cos\theta + 2d_1 r\sin\theta + \sum_{n=2}^{\infty}\left(n a_n r^{n-2} + (n+2) b_n r^n\right)\cos n\theta$$

$$+ \sum_{n=2}^{\infty}\left(n c_n r^{n-2} + (n+2) d_n r^n\right)\sin n\theta \qquad (23)$$

$$-\sum_{n=2}^{\infty}\left(a_n r^{n-2} + b_n r^n\right) n^2 \cos n\theta - \sum_{n=2}^{\infty}\left(c_n r^{n-2} + d_n r^n\right) n^2 \sin n\theta .$$

The shearing stress for the disk is found to be

$$\tau_{r\theta}' = -b_1 r\sin\theta + d_1 r\cos\theta - \sum_{n=2}^{\infty}\left(a_n r^{n-2} + b_n r^n\right) n \sin n\theta$$

$$+\sum_{n=2}^{\infty}\left(c_n r^{n-2} + d_n r^n\right) n\cos n\theta + 3b_1 r\sin\theta - 3d_1 r\cos\theta$$

$$+\sum_{n=2}^{\infty}\left(n a_n r^{n-2} + (n+2) b_n r^n\right) n\sin n\theta - \sum_{n=2}^{\infty}\left(n c_n r^{n-2} + (n+2) d_n r^n\right) n\cos n\theta$$

or

$$\tau_{r\theta}' = 2b_1 r\sin\theta - 2 d_1 r\cos\theta$$

$$+\sum_{n=2}^{\infty}\left[(n-1) a_n r^{n-2} + (n+1) b_n r^n\right] n \sin n\theta \qquad (24)$$

$$-\sum_{n=2}^{\infty}\left[(n-1) c_n r^{n-2} + (n+1) d_n r^n\right] n \cos n\theta .$$

Observation of the form of the stresses, σ_r, $\tau_{r\theta}$, in the plate reveals that these expressions contain only constants, and sines and cosines of 2θ. Thus from equations (2) the stresses, σ_r', $\tau_{r\theta}'$, in the disk must contain only constants and sines and cosines of 2θ.

Comparison of equations (20) and (23) reveals that $b_1 = d_1 = 0$ since there are no $\sin\theta$ or $\cos\theta$ terms in σ_r.

$$2b_0 = \frac{1+\alpha\epsilon}{2a^2(1+\alpha\epsilon)} A_1 + \frac{1+\beta\epsilon}{2a^2(1+\beta\epsilon)} B_1 + \frac{S}{2}$$

Therefore $$b_0 = \frac{A_1 + B_1}{4a^2} + \frac{S}{4} \qquad (25)$$

since the constants of σ_r must equal constants of σ_r'.

Comparison of the coefficients of $\cos 2\theta$ in (20) and (23) yields

$$2a_2 + 4a^2 b_2 - 4(a_2 + a^2 b_2)$$

$$= \frac{(\alpha\epsilon-1)A_1}{2a^2(\alpha\epsilon+1)} + \frac{(\beta\epsilon-1)B_1}{2a^2(\beta\epsilon+1)} + \frac{S}{2}.$$

Thus

$$a_2 = -\frac{(\alpha\epsilon-1)A_1}{4a^2(\alpha\epsilon+1)} - \frac{(\beta\epsilon-1)B_1}{4a^2(\beta\epsilon+1)} - \frac{S}{4}. \qquad (26)$$

Comparison of the coefficients of sin 2θ gives

$$2C_2 + 4a^2 d_2 - 4C_2 - 4a^2 d_2 = \frac{(\alpha\epsilon - 1)A_2}{2a^2(\alpha\epsilon + 1)} + \frac{(\beta\epsilon - 1)B_2}{2a^2(\beta\epsilon + 1)}$$

$$C_2 = -\frac{(\alpha\epsilon - 1)}{4a^2(\alpha\epsilon + 1)} A_2 - \frac{(\beta\epsilon - 1)}{4a^2(\beta\epsilon + 1)} B_2 .$$

The same procedure is followed as before. A comparison is made between equations (21) and (24). Comparing constant terms indicates that

$$0 = \frac{A_2}{2a^2} + \frac{B_2}{2a^2}$$

or

$$A_2 = -B_2 .$$

Likewise comparison of the coefficients of sin 2θ gives

$$-2d_2 - 6a^2 b_2 = -\frac{(\alpha\epsilon - 1)}{2a^2(1 + \alpha\epsilon)} A_1 - \frac{(\beta\epsilon - 1)B_1}{2a^2(\beta\epsilon + 1)} - \frac{S}{2} .$$

But the expression for a_2 is known so that

$$b_2 = \frac{1}{6a^2}\left(-\frac{(\alpha\epsilon - 1)A_1}{2a^2(\alpha\epsilon + 1)} - \frac{(\beta\epsilon - 1)B_1}{2a^2(\beta\epsilon + 1)} - \frac{S}{2} + \frac{(\alpha\epsilon - 1)A_1}{2a^2(\alpha\epsilon + 1)} + \frac{(\beta\epsilon - 1)B_1}{2a^2(\beta\epsilon + 1)} + \frac{S}{2}\right)$$

$$b_2 = 0.$$

Similarly, equating the coefficients of cos 2θ yields

$$-2C_2 - 6a^2 d_2 = -\frac{(1-\alpha\epsilon)A_2}{2a^2(1+\alpha\epsilon)} - \frac{(1-\beta\epsilon)B_2}{2a^2(1+\beta\epsilon)}.$$

Substitute here the value of c_2 obtained previously, then

$$-6a^2 d_2 = -\frac{(1-\alpha\epsilon)A_2}{a^2(1+\alpha\epsilon)} - \frac{(1-\beta\epsilon)B_2}{a^2(1+\beta\epsilon)}$$

and

$$d_2 = \frac{(1-\alpha\epsilon)A_2}{6a^4(1+\alpha\epsilon)} + \frac{(1-\beta\epsilon)B_2}{6a^4(1+\beta\epsilon)}.$$

Thus two of the boundary conditions at the periphery of the disk are satisfied, namely

$$\sigma_r'(r=a) = \sigma_r(r=a)$$

$$\tau_{r\theta}'(r=a) = \tau_{r\theta}(r=a).$$

The preceding boundary conditions have been such that all the constants of F' are zero except b_0, a_2, c_2, and d_2 so that at present the stress function for the disk is

$$F' = b_0 r^2 + a_2 r^2 \cos 2\theta + (c_2 r^2 + d_2 r^4)\sin 2\theta.$$

It is much more convenient to find the displacements in the plate in terms of the rectangular stresses σ_x and σ_y so the rectangular stresses for the disk will be obtained. This can be done very readily if r and θ are expressed directly in terms of x and y.

Thus F' becomes

$$F' = b_0(x^2+y^2) + a_2(x^2+y^2)\cos 2\theta + \left[c_2(x^2+y^2) + d_2(x^2+y^2)^2\right]\sin 2\theta$$

but $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2xy}{r^2} = \frac{2xy}{x^2+y^2}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{x^2-y^2}{x^2+y^2}$

and

$$F' = (x^2+y^2)b_0 + (x^2-y^2)a_2 + 2xyc_2 + 2xy(x^2+y^2)d_2.$$

Now obtain $\sigma_y' = \frac{\partial^2 F'}{\partial x^2}$.

$$\frac{\partial F'}{\partial x} = 2b_0x + 2a_2x + 2c_2y + d_2(6x^2y+2y^3)$$

$$\sigma_y' = \frac{\partial^2 F'}{\partial x^2} = 2b_0 + 2a_2 + 12xyd_2.$$

Next it is desired to have $\sigma_x' = \frac{\partial^2 F'}{\partial y^2}$.

$$\frac{\partial F'}{\partial y} = 2b_0y - 2a_2y + 2c_2x + (2x^3+6xy^2)d_2$$

and

$$\sigma_x' = \frac{\partial^2 F'}{\partial Y^2} = 2b_0 - 2a_2 + 12XYd_2.$$

For the problem of plane stress in an isotropic material the following relations between the components of stress and strain hold:

$$e_x = \frac{\partial U}{\partial X} = \frac{1}{E}(\sigma_x - \nu\sigma_Y)$$

$$e_Y = \frac{\partial V}{\partial Y} = \frac{1}{E}(\sigma_Y - \nu\sigma_x) \qquad (27)$$

$$\gamma_{XY} = \frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y} = \frac{1}{G}\tau_{XY}.$$

To obtain $u'(x,y)$ for the disk, integrate $\frac{\partial U'}{\partial X}$.

$$\frac{\partial U'}{\partial X} = \frac{1}{E}(\sigma_x - \nu\sigma_Y)$$

$$\frac{\partial U'}{\partial X} = \frac{1}{E}\left(2b_0 - 2a_2 + 12XYd_2 - \nu[2b_0 + 2a_2 + 12XYd_2]\right)$$

which becomes after rearrangement of terms

$$\frac{\partial U'}{\partial X} = \frac{1}{E}\left[2b_0(1-\nu) - 2a_2(1+\nu) + 12XYd_2(1-\nu)\right].$$

Hence, by integration

$$u' = \frac{1}{E}\left\{\left[2b_0(1-\nu) - 2a_2(1+\nu)\right]X + 6(1-\nu)d_2X^2Y\right\} + U_0'(Y).$$

Similarly

$$\frac{\partial v'}{\partial Y} = \frac{1}{E}(\sigma_Y - \nu\sigma_X)$$

$$\frac{\partial v'}{\partial Y} = \frac{1}{E}\left\{2b_0 + 2a_2 + 12XYd_2 - \nu(2b_0 - 2a_2 + 12XYd_2)\right\}$$

or

$$\frac{\partial v'}{\partial Y} = \frac{1}{E}\left\{2b_0(1-\nu) + 2a_2(1+\nu) + 12XYd_2(1-\nu)\right\}.$$

Integrating

$$V' = \frac{1}{E}\left\{\left[2b_0(1-\nu) + 2a_2(1+\nu)\right]Y + 6XY^2d_2(1-\nu)\right\} + V_0'(X).$$

If possible it would be desirable to determine $u_0'(y)$ and $v_0'(x)$ before proceeding further.

Using the third of equations (27) gives

$$\gamma_{XY}' = \frac{\partial v'}{\partial X} + \frac{\partial u'}{\partial Y} = \frac{1}{G'}\tau_{XY}'$$

or

$$\tau_{XY}' = G'\left(\frac{\partial v'}{\partial X} + \frac{\partial u'}{\partial Y}\right).$$

Using expressions of **u'** and **v'** above leads to

$$\tau_{XY}' = G'\left\{\frac{6(1-\nu)Y^2d_2}{E} + \frac{dv_0'}{dX} + \frac{6(1-\nu)X^2d_2}{E} + \frac{du_0'}{dY}\right\}$$

$$\tau'_{xy} = G' \left\{ \frac{6(1-\nu)(x^2+y^2)d_2}{E} + \frac{dv_o'}{dx} + \frac{du_o'}{dy} \right\}.$$

Equate this expression of τ'_{xy} to the one derived from the stress function.

It is known that

$$\tau'_{xy} = -\frac{\partial^2 F'}{\partial x \partial y}.$$

Differentiating $\frac{\partial F'}{\partial x}$ above with respect to y yields

$$\tau'_{xy} = -\frac{\partial^2 F'}{\partial x \partial y} = -2C_2 - (6x^2 + 6y^2)d_2.$$

Equating the two expressions for τ'_{xy} leads to

$$G' \left\{ \frac{6(1-\nu)(x^2+y^2)d_2}{E} + \frac{dv_o'}{dx} + \frac{du_o'}{dy} \right\} = -2C_2 - 6(x^2+y^2)d_2.$$

This requires that if d_2 is not zero that

$$\frac{6G'(1-\nu)(x^2+y^2)}{E} = -6(x^2+y^2)$$

or that

$$\frac{G(1-\nu)}{E} = 1.$$

Substituting for G gives

$$\frac{E(1-\nu)}{2(1+\nu)E} = \frac{1-\nu}{2(1+\nu)} \neq 1.$$

Therefore d_2 must equal zero. Furthermore this requires that

$$\frac{dv_o'}{dx} + \frac{du_o'}{dy} = -\frac{2C_2}{G}$$

or that

$$\frac{dv_o'}{dx} = K_4 \qquad \frac{du_o'}{dY} = K_5 \ .$$

Integrating

$$v_o' = K_4 X + K_6$$

$$u_o' = K_5 Y + K_7 \ .$$

Again no interest is found in a uniform displacement so k_6 and k_7 are set equal to zero.

Also since v must be an even function of x due to symmetry then k_4 must equal zero.

Likewise u must be an even function of y due to symmetry so k_5 must equal zero. Consequently

$$\frac{dv_o'}{dx} + \frac{du_o'}{dY} = 0 = -\frac{2C_2}{G}$$

and since G is not infinite then C_2 must equal zero.

What are the expressions for the displacements u(x,y) and v(x,y) in the plate? It is necessary to know these displacements in order that our last two boundary conditions may be applied.

Recalling the relations (3) between the stress and strain components for the two-dimensional problem of an orthotropic material and the expressions for the stresses (16), (17), and (18), the displacements may be obtained by integration

$$\frac{\partial u}{\partial x} = \frac{1}{E_x}\sigma_x - \frac{\nu_{xy}}{E_x}\sigma_y$$

$$\frac{\partial u}{\partial x} = R\left\{ A\left(\frac{\alpha^2\epsilon^2}{E_x} + \frac{\nu_{xy}}{E_x}\right)\frac{1}{w_1(z_1+w_1)} + B\left(\frac{\beta^2\epsilon^2}{E_x} + \frac{\nu_{xy}}{E_x}\right)\frac{1}{w_2(z_2+w_2)}\right\} + \frac{S}{E_x}.$$

By integration

$$u = R\left\{A\left(\frac{\alpha^2\epsilon^2+\nu_{xy}}{E_x}\right)\left(\frac{-1}{z_1+w_1}\right) + B\left(\frac{\beta^2\epsilon^2+\nu_{xy}}{E_x}\right)\left(\frac{-1}{z_2+w_2}\right)\right\} + \frac{Sx}{E_x} + u_0(y). \qquad (28)$$

Similarly

$$\frac{\partial v}{\partial y} = \frac{1}{E_y}(\sigma_y - \nu_{yx}\sigma_x)$$

$$\frac{\partial v}{\partial y} = R\left\{ A\left(\frac{1+\alpha^2\epsilon^2\nu_{yx}}{E_y}\right)\left(\frac{-1}{w_1(z_1+w_1)}\right) + B\left(\frac{1+\beta^2\epsilon^2\nu_{yx}}{E_y}\right)\left(\frac{-1}{w_2(z_2+w_2)}\right)\right\} - \frac{S\nu_{yx}}{E_y}.$$

After integration

$$v = R\left\{A\left(\frac{1+\alpha^2\epsilon^2\nu_{YX}}{E_Y}\right)\left(\frac{1}{i\alpha\epsilon(Z_1+W_1)}\right)+B\left(\frac{1+\beta^2\epsilon^2\nu_{YX}}{E_Y}\right)\left(\frac{1}{i\beta\epsilon(Z_2+W_2)}\right)\right\}$$

$$-\frac{S_Y\nu_{YX}}{E_Y}+V_0(X). \qquad (29)$$

The third equation of equations (3) is

$$\frac{\partial V}{\partial X}+\frac{\partial U}{\partial Y}=\frac{1}{G_{XY}}\tau_{XY}.$$

Taking the required partial derivatives of u and v one has

$$R\left\{A\left(\frac{1+\alpha^2\epsilon^2\nu_{YX}}{E_Y}\right)\left(\frac{-1}{i\alpha\epsilon W_1(Z_1+W_1)}\right)+B\left(\frac{1+\beta^2\epsilon^2\nu_{YX}}{E_Y}\right)\left(\frac{-1}{i\beta\epsilon W_2(Z_2+W_2)}\right)\right.$$

$$\left.+A\left(\frac{\alpha^2\epsilon^2+\nu_{XY}}{E_X}\right)\left(\frac{i\alpha\epsilon}{W_1(Z_1+W_1)}\right)+B\left(\frac{\beta^2\epsilon^2+\nu_{XY}}{E_X}\right)\left(\frac{i\beta\epsilon}{W_2(Z_2+W_2)}\right)\right\}$$

$$+\frac{dV_0(X)}{dX}+\frac{dU_0(Y)}{dY}=\frac{1}{G_{XY}}\tau_{XY}.$$

This can be rewritten as

$$\tau_{XY}=R\left\{\frac{G_{XY}A}{W_1(Z_1+W_1)}\left(\frac{i\alpha\epsilon(\alpha^2\epsilon^2+\nu_{XY})}{E_X}-\frac{(1+\alpha^2\epsilon^2\nu_{YX})}{i\alpha\epsilon E_Y}\right)\right.$$

$$+\frac{G_{xy}B}{W_2(Z_2+W_2)}\left(\frac{i\beta\epsilon(\beta^2\epsilon^2+\nu_{xy})}{E_x}-\frac{1+\beta^2\epsilon^2\nu_{yx}}{i\beta\epsilon E_y}\right)\Bigg\}+G_{xy}\left(\frac{dv_0}{dx}+\frac{du_0}{dy}\right).$$

In order to compare this expression for the shearing stress with (18) it is convenient to change the form of this expression. Rewrite τ_{xy} as

$$\tau_{xy}=R\left\{\frac{i\alpha\epsilon AG_{xy}}{W_1(Z_1+W_1)}\left(\frac{\alpha^2\epsilon^2+\nu_{xy}}{E_x}-\frac{1+\alpha^2\epsilon^2\nu_{yx}}{(i\alpha\epsilon)^2E_y}\right)\right.$$

$$\left.+\frac{i\beta\epsilon BG_{xy}}{W_2(Z_2+W_2)}\left(\frac{\beta^2\epsilon^2+\nu_{xy}}{E_x}-\frac{1+\beta^2\epsilon^2\nu_{yx}}{(i\beta\epsilon)^2E_y}\right)\right\}+G_{xy}\left(\frac{dv_0}{dx}+\frac{du_0}{dy}\right).$$

(The reader may wish to refer to pages 8 and 9 to refresh his memory on some of the relations used below.)

One has

$$G_{xy}\left(\frac{\alpha^2\epsilon^2+\nu_{xy}}{E_x}+\frac{1+\alpha^2\epsilon^2\nu_{yx}}{\alpha^2\epsilon^2E_y}\right)=\left(\frac{\alpha^2\epsilon^2+\nu_{xy}}{\epsilon^4E_y}+\frac{1+\alpha^2\epsilon^2\nu_{yx}}{\alpha^2\epsilon^2E_y}\right)G_{xy}$$

since $E_x=\epsilon^4E_y$. Also

$$\frac{\nu_{yx}}{E_y}=\frac{\nu_{xy}}{E_x}\qquad\text{or}\qquad\nu_{yx}=\frac{E_y}{E_x}\nu_{xy}=\frac{\nu_{xy}}{\epsilon^4}$$

so that the expression becomes

$$\frac{G_{xy}}{\alpha^2\epsilon^2E_y}\left(\alpha^4\epsilon^2+\alpha^2\nu_{xy}+\epsilon^2+\alpha^2\frac{\nu_{xy}}{\epsilon^4}\right)=\frac{G_{xy}}{\alpha^2\epsilon^4E_y}\left(\alpha^4\epsilon^2+\epsilon^2+2\alpha^2\nu_{xy}\right).$$

Recall that $\alpha^2\beta^2 = 1$

$$\frac{G_{xy}}{\alpha^2\epsilon^4 E_y}\left(\frac{\alpha^2\epsilon^2}{\beta^2} + \frac{\beta^2\epsilon^2}{\beta^2} + \frac{2\nu_{xy}}{\beta^2}\right) = \frac{G_{xy}}{\epsilon^4 E_y}\left(\alpha^2\epsilon^2 + \beta^2\epsilon^2 + 2\nu_{xy}\right).$$

Since

$$K = \frac{\sqrt{E_x E_y}}{2}\left(\frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x}\right) = \frac{\sqrt{E_x E_y}}{2G_{xy}} - \frac{\nu_{xy}}{\epsilon^2},$$

then

$$\frac{\sqrt{E_x E_y}}{G_{xy}} = 2K + \frac{2\nu_{xy}}{\epsilon^2} = \alpha^2 + \beta^2 + \frac{2\nu_{xy}}{\epsilon^2}$$

or

$$\frac{\alpha^2\epsilon^2 + \beta^2\epsilon^2 + 2\nu_{xy}}{\epsilon^2} = \frac{\sqrt{E_x E_y}}{G_{xy}}.$$

The expression then becomes

$$\frac{G_{xy}}{\epsilon^2 E_y}\frac{\sqrt{E_x E_y}}{G_{xy}} = \frac{1}{\epsilon^2}\sqrt{\frac{E_x}{E_y}} = \frac{\epsilon^2}{\epsilon^2} = 1.$$

Similarly

$$G_{xy}\left(\frac{\beta^2\epsilon^2 + \nu_{xy}}{E_x} - \frac{1 + \beta^2\epsilon^2\nu_{yx}}{(\alpha\beta\epsilon)^2 E_y}\right) = 1.$$

Therefore τ_{xy} can be written much more simply as

$$\tau_{xy} = R\left\{\frac{i\alpha\epsilon A}{W_1(z_1 + W_1)} + \frac{i\beta\epsilon B}{z_2(z_2 + W_2)}\right\} + G_{xy}\left(\frac{dv_0}{dx} + \frac{du_0}{dy}\right). \qquad (30)$$

Comparing (30) with (18) it is apparent that the $\tau_{r\theta}$ obtained using the stress and strain component relationships is equal to the $\tau_{r\theta}$ derived from the stress function if and only if

$$G_{xy}\left(\frac{dv_o}{dx} + \frac{du_o}{dy}\right) = 0.$$

Since G_{xy} is not equal to zero then $\frac{dv_o}{dx} + \frac{du_o}{dy}$ equals zero and

$$\frac{dv_o}{dx} = K_1$$

$$\frac{du_o}{dy} = -K_1.$$

Therefore

$$v_o = K_1 x + K_2$$

and

$$u_o = -K_1 y + K_3.$$

Since there is no interest in the case of uniform dixplacements in the x and the y directions let $k_2 = k_3 = 0$.

Consider v; it is obvious from symmetry that v must be an even function of x, that it is impossible for v to be a linear function of x. Therefore k_1 must equal zero.

Likewise u can not be an odd function of y and hence, if it were not true already, it would be true now that k_1 must equal zero.

It has been shown that $u_o(y)$ and $v_o(x)$ and that $u_o'(y)$ and $v_o'(x)$ are identically zero. Also c_2 and d_2 have been shown to equal zero.

Setting $u'(x,y)$ equal to $u(x,y)$ on the boundary of the disk gives

$$\frac{1}{E}\left\{\left(2b_0(1-\nu)-2a_2(1+\nu)\right)a\cos\theta\right\} = R\left\{\frac{-(A_1+iA_2)(\alpha^2\epsilon^2+\nu_{xy})}{a(1+\alpha\epsilon)(\cos\theta+i\sin\theta)E_x}\right.$$

$$\left. - \frac{(B_1+iB_2)(\beta^2\epsilon^2+\nu_{xy})}{a(1+\beta\epsilon)(\cos\theta+i\sin\theta)E_x}\right\} + \frac{S\,a\cos\theta}{E_x} .$$

Recalling that

$$C_2 = 0 = -\frac{\alpha\epsilon-1}{4a^2(\alpha\epsilon+1)}A_2 - \frac{\beta\epsilon-1}{4a^2(\beta\epsilon+1)}B_2$$

and that $A_2 = -B_2$, it follows that

$$\frac{\alpha\epsilon-1}{\alpha\epsilon+1}B_2 - \frac{\beta\epsilon-1}{\beta\epsilon+1}B_2 = 0.$$

This leads to

$$\left(\frac{\alpha\epsilon-1}{\alpha\epsilon+1} - \frac{\beta\epsilon-1}{\beta\epsilon+1}\right)B_2 = 0$$

and since $\frac{\alpha\epsilon-1}{\alpha\epsilon+1} - \frac{\beta\epsilon-1}{\beta\epsilon+1}$ is not equal to zero unless $\alpha\epsilon = \beta\epsilon = 0$, then

$$B_2 = 0 \text{ , and consequently}$$
$$A_2 = 0 .$$

As a result of the fact that $A_2 = B_2 = 0$ and of actually taking the real part of the expression for u it follows that

$$\frac{a}{E}\left(2b_0(1-\nu) - 2a_2(1+\nu)\right)\cos\theta = \frac{\alpha^2\epsilon^2 + \nu_{xy}}{a(1+\alpha\epsilon)E_x}\left(-A_1\cos\theta\right)$$

$$+ \frac{\beta^2\epsilon^2 + \nu_{xy}}{a(1+\beta\epsilon)E_x}\left(-B_1\cos\theta\right) + \frac{Sa\cos\theta}{E_x}.$$

Comparing the coefficients of cos θ on both sides of the equation gives

$$\frac{2a}{E}\left(b_0(1-\nu) - a_2(1+\nu)\right) = -\frac{\alpha^2\epsilon^2+\nu_{xy}}{a(1+\alpha\epsilon)E_x}A_1 - \frac{\beta^2\epsilon^2+\nu_{xy}}{a(1+\beta\epsilon)E_x}B_1 + \frac{Sa}{E_x}. \qquad (31)$$

The final boundary condition requires that for x = a cos θ, y = a sin θ that v' = v.

$$\frac{1}{E}\left\{\left[2b_0(1-\nu) + 2a_2(1+\nu)\right]a\sin\theta\right\} = R\left\{A_1\left(\frac{1+\alpha^2\epsilon^2\nu_{yx}}{E_y}\right)\left(\frac{\cos\theta - i\sin\theta}{i\alpha\epsilon a(1+\alpha\epsilon)}\right)\right.$$

$$\left. + B_1\frac{(1+\beta^2\epsilon^2\nu_{yx})(\cos\theta - i\sin\theta)}{E_y\ (i\beta\epsilon a)(1+\beta\epsilon)}\right\} - \frac{S\nu_{yx}a\sin\theta}{E_y}$$

It follows from this that

$$\frac{2a}{E}\left\{b_0(1-\nu) + a_2(1+\nu)\right\} = -\frac{(1+\alpha^2\epsilon^2\nu_{yx})}{\alpha\epsilon a(1+\alpha\epsilon)E_y}A_1 - \frac{1+\beta^2\epsilon^2\nu_{yx}}{\beta\epsilon a(1+\beta\epsilon)E_y}B_1 - \frac{S\nu_{yx}a}{E_y}. \qquad (32)$$

It is now possible to determine uniquely the constants A_1 and B_1 and in so doing to completely solve the problem.

Substitution of the values of b_0 and a_2 from equations (25) and (26) into equations (31) and (32) give two equations in the two unknowns A_1 and B_1. Since $A = A_1$ and $B = B_1$, the subscripts will no longer be carried. Thus A and B can be uniquely determined.

$$\frac{2a}{E}\left\{(1-\nu)\left(\frac{A+B}{4a^2}+\frac{S}{4}\right)+(1+\nu)\left(\frac{(\alpha\epsilon-1)A}{4a^2(\alpha\epsilon+1)}+\frac{(\beta\epsilon-1)B}{4a^2(\beta\epsilon+1)}+\frac{S}{4}\right)\right\}$$

$$= -\frac{\alpha^2\epsilon^2+\nu_{xy}}{a(1+\alpha\epsilon)E_x}A-\frac{\beta^2\epsilon^2+\nu_{xy}}{a(1+\beta\epsilon)E_x}B+\frac{Sa}{E_x}$$

Rewrite as

$$\left(\frac{1-\nu}{2aE}+\frac{(1+\nu)(\alpha\epsilon-1)}{2a(\alpha\epsilon+1)E}+\frac{\alpha^2\epsilon^2+\nu_{xy}}{a(1+\alpha\epsilon)E_x}\right)A+\left(\frac{1-\nu}{2aE}+\frac{(1+\nu)(\beta\epsilon-1)}{2a(\beta\epsilon+1)E}+\frac{\beta^2\epsilon^2+\nu_{xy}}{a(1+\beta\epsilon)E_x}\right)B$$

$$+\left(\frac{a(1-\nu)}{2E}+\frac{a(1+\nu)}{2E}-\frac{a}{E_x}\right)S=0.$$

This simplifies somewhat to

$$\left(\frac{\alpha\epsilon-\nu}{a(\alpha\epsilon+1)E}+\frac{\alpha^2\epsilon^2+\nu_{xy}}{a(1+\alpha\epsilon)E_x}\right)A+\left(\frac{\beta\epsilon-\nu}{a(\beta\epsilon+1)E}+\frac{\beta^2\epsilon^2+\nu_{xy}}{a(1+\beta\epsilon)E_x}\right)B+a\left(\frac{1}{E}-\frac{1}{E_x}\right)S=0. \quad (33)$$

Substitution into equation (32) leads to

$$\frac{2a}{E}\left\{(1-\nu)\left(\frac{A+B}{4a^2}+\frac{S}{4}\right)-(1+\nu)\left(\frac{(\alpha\epsilon-1)A}{4a^2(\alpha\epsilon+1)}+\frac{(\beta\epsilon-1)B}{4a^2(\beta\epsilon+1)}+\frac{S}{4}\right)\right\}$$

$$=-\frac{1+\alpha^2\epsilon^2\nu_{YX}}{\alpha\epsilon a(1+\alpha\epsilon)E_Y}A-\frac{1+\beta^2\epsilon^2\nu_{YX}}{\beta\epsilon a(1+\beta\epsilon)E_Y}B-\frac{S\nu_{YX}a}{E_Y}.$$

Regrouping and simplifying leads to

$$\left(\frac{1-\nu}{2aE}-\frac{(1+\nu)(\alpha\epsilon-1)}{2a(\alpha\epsilon+1)E}+\frac{1+\alpha^2\epsilon^2\nu_{YX}}{\alpha\epsilon a(1+\alpha\epsilon)E_Y}\right)A+\left(\frac{1-\nu}{2aE}-\frac{(1+\nu)(\beta\epsilon-1)}{2a(\beta\epsilon+1)E}+\frac{1+\beta^2\epsilon^2\nu_{YX}}{\beta\epsilon a(1+\beta\epsilon)E_Y}\right)B$$

$$+\left(\frac{(1-\nu)a}{2E}-\frac{(1+\nu)a}{2E}+\frac{\nu_{YX}a}{E_Y}\right)S=0$$

or

$$\left(\frac{1-\alpha\epsilon\nu}{a(1+\alpha\epsilon)E}+\frac{1+\alpha^2\epsilon^2\nu_{YX}}{\alpha\epsilon a(1+\alpha\epsilon)E_Y}\right)A+\left(\frac{1-\beta\epsilon\nu}{a(1+\beta\epsilon)E}+\frac{1+\beta^2\epsilon^2\nu_{YX}}{\beta\epsilon a(1+\beta\epsilon)E_Y}\right)B+a\left(-\frac{\nu}{E}+\frac{\nu_{YX}}{E_Y}\right)S=0. \quad (34)$$

Solution of equations (33) and (34) simultaneously yields

$$A+\frac{(\alpha\epsilon+1)\left[(\beta\epsilon-\nu)E_x+(\beta^2\epsilon^2+\nu_{XY})E\right]}{(\beta\epsilon+1)\left[(\alpha\epsilon-\nu)E_x+(\alpha^2\epsilon^2+\nu_{XY})E\right]}B=\frac{(\alpha\epsilon+1)(E-E_x)a^2S}{(\alpha\epsilon-\nu)E_X+(\alpha^2\epsilon^2+\nu_{XY})E}$$

$$A + \frac{\alpha(1+\alpha\epsilon)\left[\beta\epsilon(1-\beta\epsilon\nu)E_Y + (1+\beta^2\epsilon^2\nu_{YX})E\right]}{\beta(1+\beta\epsilon)\left[\alpha\epsilon(1-\alpha\epsilon\nu)E_Y + (1+\alpha^2\epsilon^2\nu_{YX})E\right]}B = \frac{\alpha\epsilon(1+\alpha\epsilon)(\nu E_Y - \nu_{YX}E)a^2S}{\alpha\epsilon(1-\alpha\epsilon\nu)E_Y + (1+\alpha^2\epsilon^2\nu_{YX})E}$$

$$A = \frac{\left[\frac{(E-E_X)a^2S}{(\alpha\epsilon-\nu)E_X + (\alpha^2\epsilon^2+\nu_{XY})E}\right]\left[\frac{\alpha(1+\alpha\epsilon)\left[(\beta\epsilon-\beta^2\epsilon^2\nu)E_Y + (1+\beta^2\epsilon^2\nu_{YX})E\right]}{\beta(1+\beta\epsilon)\left[(\alpha\epsilon-\alpha^2\epsilon^2\nu)E_Y + (1+\alpha^2\epsilon^2\nu_{YX})E\right]}\right]}{\frac{\alpha\left[\beta\epsilon(1-\beta\epsilon\nu)E_Y + (1+\beta^2\epsilon^2\nu_{YX})E\right]}{\beta(1+\beta\epsilon)\left[\alpha\epsilon(1-\alpha\epsilon\nu)E_Y + (1+\alpha^2\epsilon^2\nu_{YX})E\right]} - \frac{(\beta\epsilon-\nu)E_X + (\beta^2\epsilon^2+\nu_{XY})E}{(\beta\epsilon+1)\left[(\alpha\epsilon-\nu)E_X + (\alpha^2\epsilon^2+\nu_{XY})E\right]}}$$

$$- \frac{\left[\frac{\alpha\epsilon(1+\alpha\epsilon)(\nu E_Y - \nu_{YX}E)a^2S}{(\alpha\epsilon-\alpha^2\epsilon^2\nu)E_Y + (1+\alpha^2\epsilon^2\nu_{YX})E}\right]\left[\frac{(\beta\epsilon-\nu)E_X + (\beta^2\epsilon^2+\nu_{XY})E}{(\beta\epsilon+1)\left[(\alpha\epsilon-\nu)E_X + (\alpha^2\epsilon^2+\nu_{XY})E\right]}\right]}{\frac{\alpha\left[\beta\epsilon(1-\beta\epsilon\nu)E_Y + (1+\beta^2\epsilon^2\nu_{YX})E\right]}{\beta(1+\beta\epsilon)\left[\alpha\epsilon(1-\alpha\epsilon\nu)E_Y + (1+\alpha^2\epsilon^2\nu_{YX})E\right]} - \frac{(\beta\epsilon-\nu)E_X + (\beta^2\epsilon^2+\nu_{XY})E}{(\beta\epsilon+1)\left[(\alpha\epsilon-\nu)E_X + (\alpha^2\epsilon^2+\nu_{XY})E\right]}} \qquad (35)$$

$$B = \frac{\frac{\alpha\epsilon(\nu E_Y - \nu_{YX}E)a^2S}{\alpha\epsilon(1-\alpha\epsilon\nu)E_Y + (1+\alpha^2\epsilon^2\nu_{YX})E} - \frac{(E-E_X)a^2S}{(\alpha\epsilon-\nu)E_X + (\alpha^2\epsilon^2+\nu_{XY})E}}{\frac{\alpha\left[\beta\epsilon(1-\beta\epsilon\nu)E_Y + (1+\beta^2\epsilon^2\nu_{YX})E\right]}{\beta(1+\beta\epsilon)\left[\alpha\epsilon(1-\alpha\epsilon\nu)E_Y + (1+\alpha^2\epsilon^2\nu_{YX})E\right]} - \frac{(\beta\epsilon-\nu)E_X + (\beta^2\epsilon^2+\nu_{XY})E}{(\beta\epsilon+1)\left[(\alpha\epsilon-\nu)E_X + (\alpha^2\epsilon^2+\nu_{XY})E\right]}}. \qquad (36)$$

Now that the constants have been uniquely determined a listing will be made of the expressions for the stresses and the displacements in the plate and in the disk.

For the plate:

$$\sigma_X = R\left\{\frac{\alpha^2\epsilon^2 A}{W_1(z_1+W_1)} + \frac{\beta^2\epsilon^2 B}{W_2(z_2+W_2)}\right\} + S \qquad (17)$$

$$\sigma_Y = R\left\{\frac{-A}{W_1(z_1+W_1)} + \frac{-B}{W_2(z_2+W_2)}\right\} \qquad (16)$$

$$\tau_{xy} = R\left\{ \frac{i\alpha\epsilon A}{w_1(z_1+w_1)} + \frac{i\beta\epsilon B}{w_2(z_2+w_2)} \right\} \qquad (18)$$

$$u = R\left\{ A\left(\frac{\alpha^2\epsilon^2+\nu_{xy}}{E_x}\right)\left(\frac{-1}{z_1+w_1}\right) + B\left(\frac{\beta^2\epsilon^2+\nu_{xy}}{E_x}\right)\left(\frac{-1}{z_2+w_2}\right) \right\} \qquad (28)$$

$$+ \frac{S_x}{E_x}$$

$$v = R\left\{ A\left(\frac{1+\alpha^2\epsilon^2\nu_{yx}}{E_y}\right)\left(\frac{1}{i\alpha\epsilon(z_1+w_1)}\right) + B\left(\frac{1+\beta^2\epsilon^2\nu_{yx}}{E_y}\right)\left(\frac{1}{i\beta\epsilon(z_2+w_2)}\right) \right\} \qquad (29)$$

$$- \frac{S_y\nu_{yx}}{E_y}$$

For the disk:-

$$\sigma_x' = 2b_0 - 2a_2$$

$$\sigma_y' = 2b_0 + 2a_2$$

$$\tau_{xy}' = 0$$

$$u' = \frac{1}{E}\left\{ 2b_0(1-\nu) - 2a_2(1+\nu) \right\} X$$

$$v' = \frac{1}{E}\left\{ 2b_0(1-\nu) + 2a_2(1+\nu) \right\} Y$$

It is desirable to show that for a rigid disk, that is, for $E = \infty$, the constants A and B reduce to the constants obtained by Owens[10] for a rigid circular disk in an infinite orthotropic plate and thus that the solution for this condition reduces to his solution.

From (35) one obtains, using the relation $\alpha\beta = 1$, as a limit for $E \to \infty$:

$$A = \frac{\dfrac{a^2 S}{(\alpha^2\epsilon^2 + \nu_{xy})}\left(\dfrac{\alpha(1+\alpha\epsilon)(1+\beta^2\epsilon^2\nu_{yx})}{\beta(1+\beta\epsilon)(1+\alpha^2\epsilon^2\nu_{yx})}\right) + \dfrac{(1+\alpha\epsilon)(\alpha\epsilon\nu_{yx}a^2 S)(\beta^2\epsilon^2+\nu_{xy})}{(1+\alpha^2\epsilon^2\nu_{yx})(\beta\epsilon+1)(\alpha^2\epsilon^2+\nu_{xy})}}{\dfrac{\alpha(1+\beta^2\epsilon^2\nu_{yx})}{\beta(1+\beta\epsilon)(1+\alpha^2\epsilon^2\nu_{yx})} - \dfrac{\beta^2\epsilon^2+\nu_{xy}}{(1+\beta\epsilon)(\alpha^2\epsilon^2+\nu_{xy})}}.$$

Combining the denominator into a single fraction this becomes

$$A = \frac{\dfrac{a^2 S}{(\alpha^2\epsilon^2+\nu_{xy})}\left(\dfrac{\alpha(1+\alpha\epsilon)(1+\beta^2\epsilon^2\nu_{yx})}{\beta(1+\beta\epsilon)(1+\alpha^2\epsilon^2\nu_{yx})}\right) + \dfrac{\beta(1+\alpha\epsilon)(\alpha\epsilon\nu_{yx}a^2 S)(\beta^2\epsilon^2+\nu_{xy})}{(1+\alpha^2\epsilon^2\nu_{yx})(1+\beta\epsilon)(\alpha^2\epsilon^2+\nu_{xy})}}{\dfrac{\alpha(1+\beta^2\epsilon^2\nu_{yx})(\alpha^2\epsilon^2+\nu_{xy}) - \beta(\beta^2\epsilon^2+\nu_{xy})(1+\alpha^2\epsilon^2\nu_{yx})}{\beta(1+\beta\epsilon)(1+\alpha^2\epsilon^2\nu_{yx})(\alpha^2\epsilon^2+\nu_{xy})}}.$$

This reduces to

$$A = \frac{a^2 S\left\{\alpha(1+\alpha\epsilon)(1+\beta^2\epsilon^2\nu_{yx}) + \epsilon\nu_{yx}(1+\alpha\epsilon)(\beta^2\epsilon^2+\nu_{xy})\right\}}{\alpha(1+\beta^2\epsilon^2\nu_{yx})(\alpha^2\epsilon^2+\nu_{xy}) - \beta(\beta^2\epsilon^2+\nu_{xy})(1+\alpha^2\epsilon^2\nu_{yx})}$$

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[10] Owens, op. cit., p. 32.

which is the value for A obtained by Owens for a circular disk of radius a.

Equation (36) gives for $E \to \infty$

$$B = \frac{-\frac{\alpha\epsilon\nu_{yx}(1+\alpha\epsilon)a^2S}{1+\alpha^2\epsilon^2\nu_{yx}} - \frac{\alpha\epsilon+1}{\alpha^2\epsilon^2+\nu_{xy}}\,a^2S}{\frac{\alpha(1+\alpha\epsilon)(1+\beta^2\epsilon^2\nu_{yx})}{\beta(1+\beta\epsilon)(1+\alpha^2\epsilon^2\nu_{yx})} - \frac{(1+\alpha\epsilon)(\beta^2\epsilon^2+\nu_{xy})}{(1+\beta\epsilon)(\alpha^2\epsilon^2+\nu_{xy})}}$$

or

$$B = \frac{-a^2S\left(\frac{\alpha\epsilon\nu_{yx}(1+\alpha\epsilon)(\alpha^2\epsilon^2+\nu_{xy})+(1+\alpha\epsilon)(1+\alpha^2\epsilon^2\nu_{yx})}{(1+\alpha^2\epsilon^2\nu_{yx})(\alpha^2\epsilon^2+\nu_{xy})}\right)}{\frac{\alpha(1+\alpha\epsilon)(1+\beta^2\epsilon^2\nu_{yx})(\alpha^2\epsilon^2+\nu_{xy})-\beta(1+\alpha\epsilon)(\beta^2\epsilon^2+\nu_{xy})(1+\alpha^2\epsilon^2\nu_{yx})}{\beta(1+\beta\epsilon)(1+\alpha^2\epsilon^2\nu_{yx})(\alpha^2\epsilon^2+\nu_{xy})}}.$$

The expression for B simplifies to

$$B = -(1+\beta\epsilon)a^2S\left\{\frac{\epsilon\nu_{yx}(\alpha^2\epsilon^2+\nu_{xy})+\beta(1+\alpha^2\epsilon^2\nu_{yx})}{\alpha(1+\beta^2\epsilon^2\nu_{yx})(\alpha^2\epsilon^2+\nu_{xy})-\beta(\beta^2\epsilon^2+\nu_{xy})(1+\alpha^2\epsilon^2\nu_{yx})}\right\}.$$

This is the same value for B as obtained by Owens and thus the solution found above becomes the solution obtained by Owens when $E = \infty$. It has thus been shown that Owens' problem is a special case of the above problem, namely whenever $E = \infty$.

Furthermore it is desired to show that the problem of this dissertation becomes the problem solved by Professor Smith[11]. It is recalled that Professor Smith solved the

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[11] Smith, op. cit.

problem of an infinite orthotropic plate with an elliptic or circular hole at the center. The equality of the two solutions will be shown by demonstrating that the A and B of the above problem become the A and B of Professor Smith's problem (a = b) whenever E = 0.

Equation (35) gives for E = 0

$$A = \frac{\dfrac{-a^2S}{(\alpha\epsilon-\nu)}\left\{\dfrac{\alpha(1+\alpha\epsilon)(\beta\epsilon-\beta^2\epsilon^2\nu)}{\beta(1+\beta\epsilon)(\alpha\epsilon-\alpha^2\epsilon^2\nu)}\right\} - \dfrac{\alpha\epsilon(1+\alpha\epsilon)\nu a^2S(\beta\epsilon-\nu)}{(\alpha\epsilon-\alpha^2\epsilon^2\nu)(1+\beta\epsilon)(\alpha\epsilon-\nu)}}{\dfrac{(1-\beta\epsilon\nu)}{(1+\beta\epsilon)(1-\alpha\epsilon\nu)} - \dfrac{(\beta\epsilon-\nu)}{(1+\beta\epsilon)(\alpha\epsilon-\nu)}}.$$

This can be written as

$$A = -(1+\alpha\epsilon)a^2S\,\frac{\dfrac{\alpha\beta\epsilon(1-\beta\epsilon\nu)}{\beta(\alpha\epsilon-\nu)\alpha\epsilon(1-\alpha\epsilon\nu)} + \dfrac{\alpha\epsilon\nu(\beta\epsilon-\nu)}{\alpha\epsilon(1-\alpha\epsilon\nu)(\alpha\epsilon-\nu)}}{\dfrac{(1-\beta\epsilon\nu)(\alpha\epsilon-\nu) - (\beta\epsilon-\nu)(1-\alpha\epsilon\nu)}{(1-\alpha\epsilon\nu)(\alpha\epsilon-\nu)}}.$$

Regrouping, one obtains

$$A = -(1+\alpha\epsilon)a^2S\left\{\frac{1-\nu^2}{\alpha\epsilon-\beta\epsilon-\alpha\epsilon\nu^2+\beta\epsilon\nu^2}\right\}$$

which reduces to

$$A = -\frac{(1+\alpha\epsilon)a^2S}{\epsilon(\alpha-\beta)}.$$

This is precisely the A obtained by Professor Smith for the free circular hole.

From equation (36) for E = 0 one has

$$B = \frac{\dfrac{(1+\alpha\epsilon)\,\nu a^2 S}{1-\alpha\epsilon\nu} + \dfrac{(\alpha\epsilon+1)\,a^2 S}{\alpha\epsilon-\nu}}{\dfrac{(1+\alpha\epsilon)(1-\beta\epsilon\nu)}{(1+\beta\epsilon)(1-\alpha\epsilon\nu)} - \dfrac{(1+\alpha\epsilon)(\beta\epsilon-\nu)}{(1+\beta\epsilon)(\alpha\epsilon-\nu)}}$$

or

$$B = (1+\beta\epsilon)a^2 S\ \frac{\nu(\alpha\epsilon-\nu)+1-\alpha\epsilon\nu}{(1-\beta\epsilon\nu)(\alpha\epsilon-\nu)-(\beta\epsilon-\nu)(1-\alpha\epsilon\nu)}$$

This can be written as

$$B = (1+\beta\epsilon)a^2 S\ \frac{1-\nu^2}{\alpha\epsilon-\beta\epsilon-\alpha\epsilon\nu^2+\beta\epsilon\nu^2}$$

Therefore

$$B = \frac{(1+\beta\epsilon)a^2 S}{\epsilon(\alpha-\beta)}$$

which is the value obtained by Professor Smith for B. Consequently, the problem of this dissertation becomes the problem treated by Professor Smith when a = b and when E = 0.

CHAPTER III

THE ISOTROPIC STRESS FUNCTION OBTAINED AS THE LIMITING CASE OF THE ORTHOTROPIC STRESS FUNCTION

The stress function for an infinite isotropic plate containing a circular isotropic disk at its center will be obtained as the limiting case of the orthotropic stress function (14), where A and B of equation (14) are given by equations (35) and (36) respectively.

It is a necessary, but not a sufficient, condition for a material to be isotropic that $E_x = E_y$. For some orthotropic materials it is true that $E_x \doteq E_y$. For convenience it is assumed that this equality holds here since this assumption causes no loss of generality. Thus

$$\epsilon = \sqrt[4]{\frac{E_x}{E_y}} = 1 .$$

Recall that $\alpha^2 = e^{\Phi}$, $\beta^2 = e^{-\Phi}$ and let

$$C = \frac{E - E_x}{(\alpha - \nu)E_x + (\alpha^2 + \nu_{xy})E} = \frac{E' - E}{(e^{\Phi/2} - \nu')E + (e^{\Phi} + \nu)E'}$$

$$D = \frac{\alpha(1+\alpha)}{\beta(1+\beta)}\left[\frac{(\beta - \beta^2\nu)E_y + (1 + \beta^2\nu_{yx})E}{(\alpha - \alpha^2\nu)E_y + (1 + \alpha^2\nu_{yx})E}\right]$$

$$D = \frac{e^{\Phi/2}(1+e^{\Phi/2})}{e^{-\Phi/2}(1+e^{-\Phi/2})}\left[\frac{(e^{-\Phi/2} - e^{-\Phi}\nu')E + (1+e^{-\Phi}\nu)E'}{(e^{\Phi/2} - e^{\Phi}\nu')E + (1+e^{\Phi}\nu)E'}\right]$$

$$G = \frac{\alpha(1+\alpha)(\nu E_Y - \nu_{YX}E)}{(1+\beta)[(\alpha-\alpha^2\nu)E_Y + (1+\alpha^2\nu_{YX})E]}$$

$$G = \frac{e^{\Phi/2}(1+e^{\Phi/2})(\nu'E - \nu E')}{(1+e^{-\Phi/2})[(e^{\Phi/2}-e^{\Phi}\nu')E + (1+e^{\Phi}\nu)E']}$$

$$H = \frac{(\beta-\nu)E_X + (\beta^2+\nu_{XY})E}{(\alpha-\nu)E_X + (\alpha^2+\nu_{XY})E}$$

$$H = \frac{(e^{-\Phi/2}-\nu')E + (e^{-\Phi}+\nu)E'}{(e^{\Phi/2}-\nu')E + (e^{\Phi}+\nu)E'}$$

$$I = \frac{\alpha(\nu E_Y - \nu_{YX}E)}{\alpha(1-\alpha\nu)E_Y + (1+\alpha^2\nu_{YX})E} = \frac{e^{\Phi/2}(\nu'E - \nu E')}{e^{\Phi/2}(1-e^{\Phi/2}\nu')E + (1+e^{\Phi}\nu)E'}$$

$$J = \frac{E-E_X}{(\alpha-\nu)E_X + (\alpha^2+\nu_{XY})E} = \frac{E-E_X}{(e^{\Phi/2}-\nu')E + (e^{\Phi}+\nu)E'}$$

$$K = \frac{\alpha[(\beta)(1-\beta\nu)E_Y + (1+\beta^2\nu_{YX})E]}{\beta(1+\beta)[\alpha(1-\alpha\nu)E_Y + (1+\alpha^2\nu_{YX})E]} - \frac{(\beta-\nu)E_X + (\beta^2+\nu_{XY})E}{(1+\beta)[(\alpha-\nu)E_X + (\alpha^2+\nu_{XY})E]}$$

$$K = \frac{e^{\Phi/2}[e^{-\Phi/2}(1-e^{-\Phi/2}\nu')E + (1+e^{-\Phi}\nu)E']}{e^{-\Phi/2}(1+e^{-\Phi/2})[e^{\Phi/2}(1-e^{\Phi/2}\nu')E + (1+e^{\Phi}\nu)E']} - \frac{(e^{-\Phi/2}\nu')E + (e^{-\Phi}+\nu)E'}{(1+e^{-\Phi/2})[(e^{\Phi/2}\nu')E + (e^{\Phi}+\nu)E']}$$

$$f(z_1) = \left[\tfrac{1}{2}(z_1-w_1)^2 + \gamma_1^2 \ln(z_1+w_1)\right]$$

$$g(z_2) = \left[\tfrac{1}{2}(z_2-w_2)^2 + \gamma_2^2 \ln(z_2+w_2)\right]$$

where

$$z_1 = x + i\alpha\epsilon Y = x + ie^{\Phi/2}\epsilon Y$$

$$z_2 = x + i\beta\epsilon Y = x + ie^{-\Phi/2}\epsilon Y.$$

The stress function F can then be written as

$$F(\phi) = \frac{S}{2} R\left\{ \frac{CD-GH}{(1-e^{\Phi})K} f[z_1(\phi)] + \frac{I-J}{(1-e^{-\Phi})K} g[z_2(\phi)] \right\} + \frac{Sy^2}{2}.$$

Express F as follows

$$F(\phi) = \frac{S}{2} R\left\{ \frac{(1-e^{-\Phi})(CD-GH) f[z_1(\phi)] + (1-e^{\Phi})(I-J) g[z_2(\phi)]}{(1-e^{\Phi})(1-e^{-\Phi})K} \right\} + \frac{Sy^2}{2}.$$

For an isotropic material Φ becomes equal to zero[12]. As $\Phi \to 0$, F becomes indeterminate and assumes the form

$$F = \frac{S}{2} R\left\{ \frac{0}{0} \right\} + \frac{Sy^2}{2}.$$

Let

$$N = (1-e^{-\Phi})(CD-GH) f[z_1(\phi)] + (1-e^{\Phi})(I-J) g[z_2(\phi)]$$

$$W = (1-e^{\Phi})(1-e^{-\Phi}) K.$$

Using L'Hospital's rule to ascertain the value of F as $\Phi \to 0$ one finds that

$$\lim_{\Phi \to 0} N' = \lim_{\Phi \to 0} N'' = \lim_{\Phi \to 0} W' = \lim_{\Phi \to 0} W'' = 0$$

where the prime designates the derivative with respect to ϕ.

One obtains as a limit after straightforward but somewhat tedious differentiation

$$\lim_{\Phi \to 0} \frac{d^3W}{d\phi^3} = - \frac{3[(1+\nu')E + (3-\nu)E']}{(1-\nu')E + (1+\nu)E'}.$$

---

[12] Since $\Phi = \cosh^{-1} \sqrt{\frac{E_x E_y}{2}} \left(\frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x} \right)$

Similarly one obtains as a limit

$$\lim_{\varphi \to 0} \frac{d^3N}{d\varphi^3} = -3a^2 \left\{ \frac{(2E'+E)[(1+\nu')E-(1+\nu)E']}{[(1-\nu')E+(1+\nu)E']^2} \right.$$

$$\left. + \frac{(E'-E)[(2+\nu')E+(5-\nu)E'] + (\nu'E-\nu E')[\nu'E+(1-\nu)E']}{[(1-\nu')E+(1+\nu)E']^2} \right\} \ln z\bar{z}$$

$$- \frac{6[E'-E-\nu'E+\nu E']}{(1-\nu')E+(1+\nu)E'} \left(\frac{ia^2\gamma}{z} + \frac{a^4}{4z^2} \right).$$

Therefore the stress function for the isotropic plate with a circular disk of a different isotropic material at its center is

$$F = \frac{S}{2} R \left\{ \left[\frac{(2E'+E)[(1+\nu')E-(1+\nu)E'] + (E'-E)[(2+\nu')E+(5-\nu)E']}{[(1-\nu')E+(1+\nu)E'][(1+\nu')E+(3-\nu)E']} \right.\right.$$

$$\left. + \frac{(\nu'E-\nu E')[\nu'E+(1-\nu)E']}{[(1-\nu')E+(1+\nu)E'][(1+\nu')E+(3-\nu)E']} \right] a^2 \ln z\bar{z}$$

$$\left. + \frac{2[(1+\nu)E'-(1+\nu')E]}{(1+\nu')E+(3-\nu)E'} \left(\frac{ia^2\gamma}{z} + \frac{a^4}{4z^2} \right) \right\} + \frac{S\gamma^2}{2}.$$

APPENDIX

For a numerical example to illustrate the theory previously developed, an infinite Sitka spruce plate with a circular copper disk at its center subjected to a uniform tension in the direction of the grain along opposite edges of the plate will be considered. The X-axis is taken in the direction of the grain and the Y-axis is taken to be perpendicular to the grain. The normal stress components in the plate along the two axes exterior to the copper disk, and the shearing stress components along the boundary of the disk are obtained.

The same Sitka spruce plate subjected to the same uniform tension as above but with a circular rubber disk at its center is considered. The same stresses as above are obtained.

For an example of an infinite isotropic plate a steel plate is chosen. It also is subjected to a uniform tension along opposite edges as previously and contains a circular copper disk at its center. The same plate is considered with a circular rubber disk at its center. In both cases the same stresses as found above are obtained.

The tables and the graphs are self-explanatory.

TABLE I

ELASTIC MODULI OF SITKA SPRUCE*

ν_{xy}	0.464
E_x	1,679,000 lbs. per sq. in.
E_y	76,000 lbs. per sq. in.
G_{xy}	112,000 lbs. per sq. in.

*The elastic moduli of Sitka spruce are average values obtained from a large number of tests made at the U. S. Forest Products Laboratory.

TABLE II

CONSTANTS EVALUATED FOR SITKA SPRUCE

ν_{yx}	0.0210	$\alpha^2\epsilon^2$	12.26
ϵ	2.168	$\beta\epsilon$	1.342
ϵ^2	4.700	$\beta^2\epsilon^2$	1.802
K	1.496	$1+\alpha^2\epsilon^2\nu_{yx}$	1.257
α	1.615	$\alpha^2\epsilon^2+\nu_{xy}$	12.72
α^2	2.608	$1+\beta^2\epsilon^2\nu_{yx}$	1.0378
β	0.6192	$\beta^2\epsilon^2+\nu_{xy}$	2.266
β^2	0.3834	$1+\alpha\epsilon$	4.501
$\alpha\epsilon$	3.501	$1+\beta\epsilon$	2.342

TABLE III

ELASTIC CONSTANTS FOR COPPER, HARD RUBBER, AND STEEL

	E	ν
COPPER	15,000,000 lbs. per sq. in.	0.33
HARD RUBBER	4,000 " " " "	0.25
STEEL	29,500,000 " " " "	0.29

TABLE IV

CONSTANTS A AND B EVALUATED FOR AN INFINITE PLAIN-SAWN SITKA SPRUCE PLATE CONTAINING A CIRCULAR ISOTROPIC DISK OF RADIUS a AT ITS CENTER

	A	B
COPPER DISK	$0.3561\ a^2S$	$-0.1439\ a^2S$
HARD RUBBER DISK	$-2.0394\ a^2S$	$1.0429\ a^2S$

TABLE V

VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE X-AXIS EXTERIOR TO A CIRCULAR COPPER DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

X	σ_x/S	σ_y/S
a	1.1945	0.0232
2a	1.1610	0.0002
3a	1.1158	- 0.0030
4a	1.0829	- 0.0031
5a	1.0607	- 0.0026

TABLE VI

VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE Y-AXIS EXTERIOR TO A CIRCULAR COPPER DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

Y	σ_x/S	σ_Y/S
a	0.1409	0.0177
1.05a	0.5342	
1.10a	0.6812	
1.20a	0.8091	
1.50a	0.9214	
2a	0.9656	- 0.0065
3a	0.9869	- 0.0029
4a	0.9930	- 0.0016
5a	0.9956	- 0.0010

TABLE VII

VALUES OF THE SHEAR STRESS COMPONENTS AT POINTS ALONG THE BOUNDARY OF A CIRCULAR COPPER DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

θ in degrees	τ_{xy}/S
0	0
10	- 0.0007
20	0.0003
30	0.0050
40	0.0160
50	0.0372
60	0.0719
70	0.1162
80	0.1269
85	0.0831
90	0

TABLE VIII

VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE X-AXIS EXTERIOR TO A CIRCULAR HARD RUBBER DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

X	$\sigma_{x/s}$	$\sigma_{y/s}$
a	0.0115	- 0.2024
2a	0.1210	- 0.0252
3a	0.3574	0.0061
4a	0.5373	0.0110
5a	0.6600	0.0104

TABLE IX

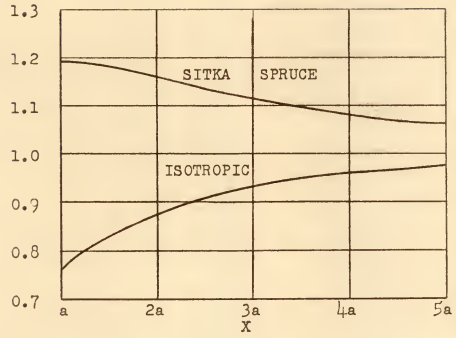
VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE Y-AXIS EXTERIOR TO A CIRCULAR HARD RUBBER DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

Y	$\sigma_{x/s}$	$\sigma_{y/s}$
a	5.7517	- 0.0078
1.05a	3.5225	0.1216
1.10a	2.6991	0.1477
1.20a	1.9931	0.1445
1.50a	1.3928	0.0972
2a	1.1669	0.0538
3a	1.0626	0.0234
4a	1.0333	0.0130
5a	1.0208	0.0083

TABLE X

VALUES OF THE SHEAR STRESS COMPONENTS AT POINTS ALONG THE BOUNDARY OF A CIRCULAR HARD RUBBER DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

θ in degrees	τ_{xy}/s
0	0
10	0.0322
20	0.0536
30	0.0510
40	0.0064
50	- 0.1045
60	- 0.3057
70	- 0.5792
80	- 0.6778
85	- 0.4503
90	0



$\frac{\sigma_x}{q}$

Figure 4.-- Variation of the normal stress component σ_x at points along the X-axis exterior to a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

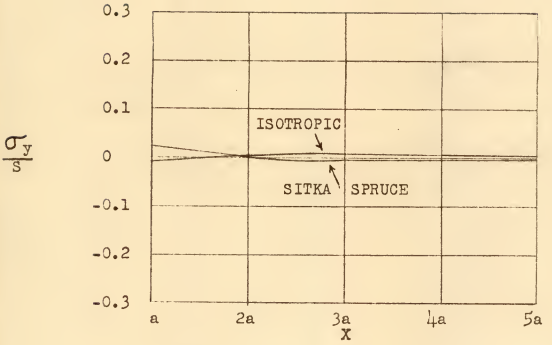
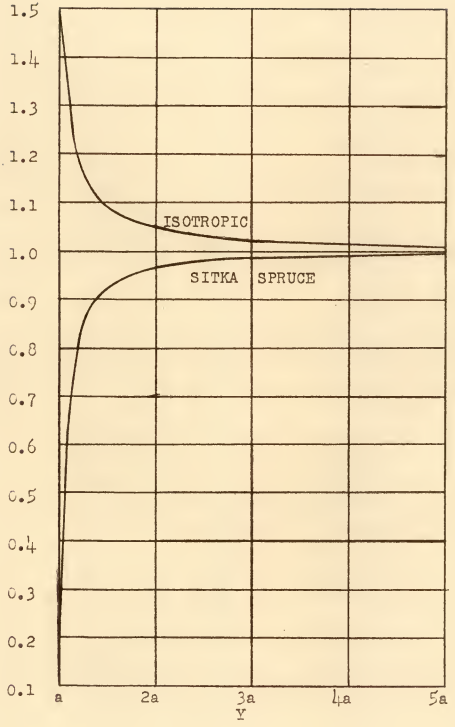


Figure 5.-- Variation of the normal stress component σ_Y at points along the X-axis exterior to a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.



$\frac{\sigma_x}{s}$

Figure 6.-- Variation of the normal stress component σ_x at points along the Y-axis exterior to a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

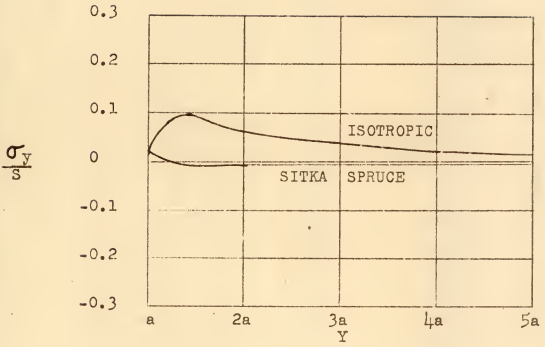


Figure 7.-- Variation of the normal stress component σ_Y at points along the Y-axis exterior to a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

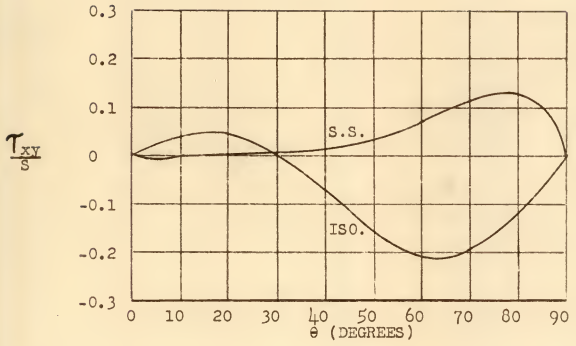


Figure 8.--Variation of the shear stress component τ_{xy} at points along the boundary of a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

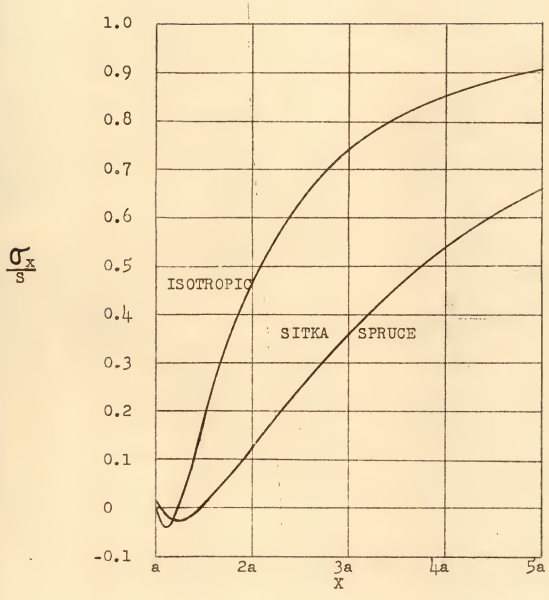


Figure 9.-- Variation of the normal stress component σ_x at points along the X-axis exterior to a circular rubber disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

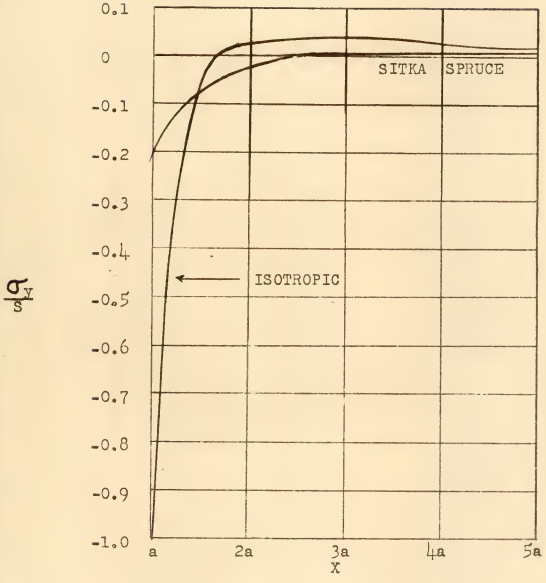


Figure 10.-- Variation of the normal stress component σ_y at points along the X-axis exterior to a circular rubber disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

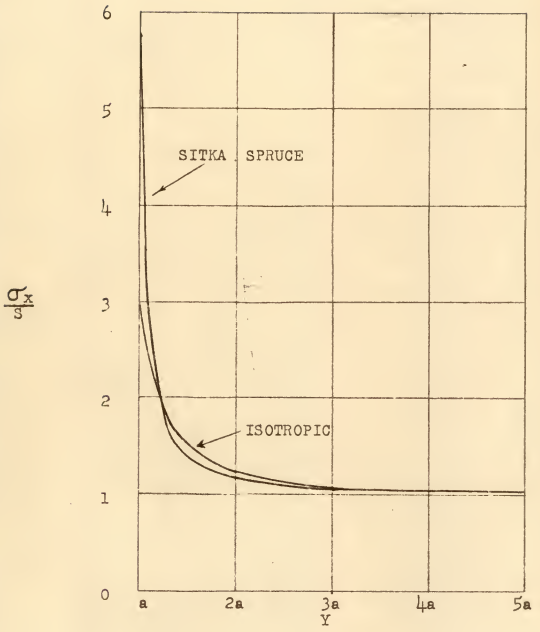


Figure 11.-- Variation of the normal stress component σ_x at points along the Y-axis exterior to a circular rubber disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

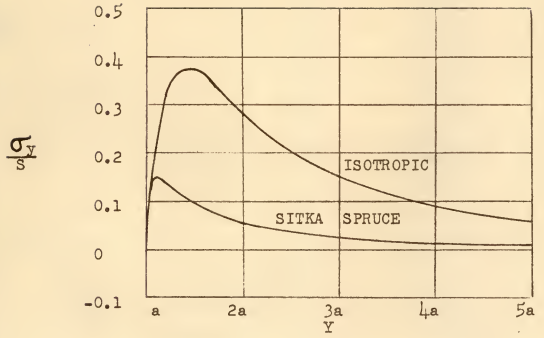


Figure 12.-- Variation of the normal stress component σ_Y at points along the Y-axis exterior to a circular rubber disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

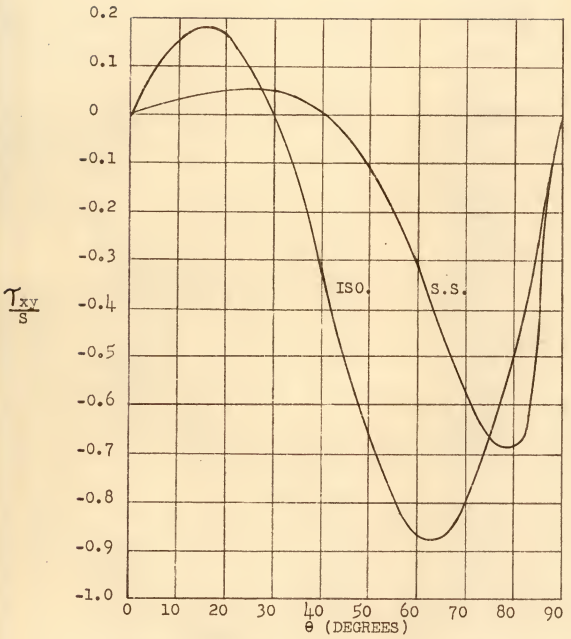


Figure 13.--Variation of the shear stress component τ_{xy} at points along the boundary of a circular rubber disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

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BIOGRAPHY

The author was born November 25, 1921 at Plant City, Florida. In 1939 he graduated from Plant City High School, Plant City, Florida. After one year at Florida Southern College he attended the University of Florida. In January, 1943 he entered the Air Force in which he served 3½ years, during which time he gained the rank of Captain. After his release from active duty in June, 1946 he returned to the University of Florida from which he received his Bachelor of Science degree in February, 1947. In the fall of 1946 the author became a member of Phi Beta Kappa. After some graduate work at the University, he was a Special Agent of the Federal Bureau of Investigation for two years. He returned to the University again in June, 1950 to complete work on his advanced degrees.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

August 16, 1952

W. O. Rose
Dean of the College of Arts and Sciences

C. B. Smith
Dean of the Graduate School

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